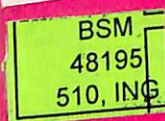


B.Sc. Second Year

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

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Dr. R. N. Ingle



Aruna Prakashan

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

B.Sc second year

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PREFACE

There are many books on the well-known topic of differential equations. Some of these books focus on theoretical aspects whereas other give techniques of solving the problems. There has been a need to have both their theoretical development and problem solving skills. Taking this need into consideration, we have written this book.

The main aim of this book is to present the elements of differential equations in a form suitable for the use of undergraduate students of mathematics, with emphasis on developing problem solving skills but nevertheless supported by adequate theory without going into the intricacies of some involved proofs, book included chapter s laplace and fourier transforms has the advantages of directly giving the solution of differential equations. In its present form, the book has been developed from a course offered by the authors to the B.Sc. students of S.R.T. Marathwada University Nanded and other Universities in maharashtra.

Chapter - 1, Introduces the some basic concept of partial differential equations, Methods of forming partial differential equations, and Solution of equations by direct integration, Lagarangean linear equation are discussed with illustrative examples.

Chapter - 2, In this chapter partial differential equations Non-linear in p & q are discussed and various methods of solving partial differential equation are also seen with examples.

Chapter - 3, In thischapter We discussed pratical problems of wave equation, one dimensional heat flow, two dimensional heat flow, radio equation.

chapter-4&5, Both chapters we discussed integral transforms like Laplace and fourier transforms and their properties.,inverse transforms,transforms of special functionsand convolution theorem.

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PARTIAL DIFFERENTIAL EQUATIONS

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Index

CHAPTER - 1 PARTIAL DIFFERENTIAL EQUATIONS

1 - 44

- 1.1 Introduction
- 1.2 Partial Differential Equations (P.D.E). .. 48195
- 1.3 Order of A Partial Differential Equation
- 1.4 Degree of A Partial Differential Equation
- 1.5 Linear And Non-Linear Partial Differential Equations
- 1.6 Notations
- 1.7 Method of Forming Partial Differential Equations
- 1.8 Solution of Equation By Direct Integration
- 1.9 Lagrange's Linear Equation Is An Equation of The Type
- + ~ Multiple Choice Questions

CHAPTER - 2

45 - 103

PARTIAL DIFFERENTIAL EQUATIONS NON-LINEAR IN p AND q

- 2.1 Introduction
- 2.2 Charpit's Method
- 2.3 Linear Homogenous Partial Differential Equations of Order With Constant Coefficients

- 2.4 Rules For Finding The Particular Integral
- 2.5 Non-Homogeneous Linear Equations
- 2.6 Particular Integral
- 2.7 Monges Method (Non Linear Equation of The Second Order)
- + Multiple Choice Questions

CHAPTER - 3

PARTIAL DIFFERENTIAL EQUATIONS IN PRACTICAL PROBLEM

104 - 152

- 3.1 Introduction
- 3.2 Method of Separation of Variables
- 3.3 Equation of Vibrating String
- 3.4 Solution of Wave Equation By D'almbert's Method
- 3.5 One Dimensional Heat Flow
- 3.6 Two Dimensional Heat Flow
- 3.7 Laplace Equation In Polar Co-Ordinates
- 3.8 Transmission Line Equations
- + Multiple Choice Questions

CHAPTER - 4

LAPLACE TRANSFORMS

153 - 193

- 4.1 Introduction
- 4.2 Definition of laplace transform
- 4.3 Transform sof basic functions
- 4.4 Basic Properties of laplace transforms
- 4.5 periodic functions
- 4.6 special functions
- 4.7 Derivatives
- 4.8 Integrals
- 4.9 multiplication by tn
- 4.10 Division by t
- 4.11 Evaluation of integral by laplace transform
- 4.12 Inverse transforms method by partial fractions
- 4.13 Other methods of finding Inverse transform
- 4.14 Convolution Theoreom
- + Multiple Choice Questions

CHAPTER - 5

FOURIER TRANSFORMS

194 - 222

- 5.1 Introduction

- 5.2 Definition of fourier transform
- 5.3 Fourier integral theorem
- 5.4 Fourier transforms
- 5.5 Properties of Fourier transforms
- 5.6 Convolution
- 5.6 Parseval's Theorem for Fourier transforms
- + Multiple Choice Questions

References

223

1

PARTIAL DIFFERENTIAL EQUATIONS

1.1 INTRODUCTION :-

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E) :-

Definition:- An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.

For example of partial differential we list the following.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \quad \dots\dots(1)$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \quad \dots\dots(2)$$

$$z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x \quad \dots\dots(3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz \quad \dots\dots(4)$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \quad \dots\dots(5)$$

$$y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right) \quad \dots\dots(6)$$

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION :-

Definition:- The order of a partial differential equation is defined as the order of the highest partial derivatives occurring in the partial differential equation.

In Art 1.2 equation (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION :-

The degree of a partial differential equation is the degree of the highest order derivatives which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS :-

Definition:- A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called as a non-linear partial differential equation.

In Art. 1.2 equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS :-

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \partial z / \partial x, \quad q = \partial z / \partial y, \quad r = \partial^2 z / \partial x^2, \quad s = \partial^2 z / \partial x \partial y \quad \text{and} \quad t = \partial^2 z / \partial y^2$$

1.7 METHOD OF FORMING PARTIAL DIFFERENTIAL EQUATIONS :-

A partial differential equation is formed by two methods

- (i) By eliminating arbitrary constants
- (ii) By eliminating arbitrary functions
- (i) By eliminating arbitrary constants

Example 1. Form a partial differential equation from

$$x^2 + y^2 + (z - c)^2 = a^2$$

Solution:- $x^2 + y^2 + (z - c)^2 = a^2$... (1)

Contains two arbitrary constants a and c .

Differentiating (1) partially w.r.t. x we get

$$2x + 2(z - c) \frac{\partial z}{\partial x} = 0$$

$$x + (z - c)p = 0 \quad \dots (2)$$

Differentiating (1) partially w.r.t. y we get

$$2y + 2(z - c) \frac{\partial z}{\partial y} = 0$$

$$y + (z - c)q = 0 \quad \dots (3)$$

Let us eliminate C from (2) and (3)

$$\text{From (2)} \quad (z - c) = -\frac{x}{p}$$

Putting this value of $z - c$ in (3), we get $y - \frac{x}{p}q = 0$

$$\text{or} \quad yp - xq = 0 \quad \text{Ans.}$$

Example 2 : $(x - h)^2 + (y - k)^2 + z^2 = a^2$

Solution : We have

$$(x - h)^2 + (y - k)^2 + z^2 = a^2 \quad \dots (1)$$

Differentiating equation (1) partially w.r.t. x and y , we get

$$x - h + zp = 0 \quad \text{and} \quad y - k + zq = 0$$

Putting the values of $x-h$ and $y-k$ from the last two equations in the given equation (1), we get $z^2(p^2 + q^2 + 1) = a^2$

Which is the required P.D.E.

Example 3: $z = (x+a)(y+b)$

Solution: We have

$$z = (x+a)(y+b) \quad \dots(1)$$

Differentiating equation (1) partially w.r.t x and y we get

$$\frac{\partial z}{\partial x} = p = (y+b) \quad \dots(2)$$

$$\text{and } \frac{\partial z}{\partial y} = q = (x+a) \quad \dots(3)$$

Putting the values, of $(y+b)$ and $(x+a)$ from (2) and (3), we get in the equation (1), we get

$z = pq$, which is the required P.D.E.

Example 4: $x^2 + y^2 = (z-c)^2 \tan^2 \alpha$

Solution: Differential partially w.r.t x & y , we get

$$2x = 2(z-c)p \tan^2 \alpha \frac{dz}{dx} \quad (i)$$

$$2y = 2(z-c)q \tan^2 \alpha \frac{dz}{dy} \quad (ii)$$

From (i) & (ii) eliminate contract C

$$\frac{2x}{p \tan^2 \alpha} = 2(z-c)$$

$$(z-c) = \frac{x}{p \tan^2 \alpha} \text{ put in}$$

(iii)

$$y = (z-c)q \tan^2 \alpha$$

$$y = \frac{x}{p \tan^2 \alpha} \cdot q \tan^2 \alpha$$

$$yp = xq$$

$$yp - xq = 0$$

(ii) **Method of elimination of arbitrary function**

Example 5: $z = f(x^2 + y^2)$

Solution: We have

$$z = f(x^2 + y^2) \quad \dots(1)$$

Diff. (1) partially w.r.t x and y , we get

$$p = f'(x^2 + y^2) \cdot 2x \quad \dots(2)$$

$$q = f'(x^2 + y^2) \cdot 2y \quad \dots(3)$$

From (2) and (3), we get

$py - qx = 0$, which is the required P.D.E.

Example 6: $f(x + y + z, x^2 + y^2 + z^2) = 0$

Solution: We have,

$$f(x + y + z, x^2 + y^2 + z^2) = 0 \quad \dots(1)$$

Let, $u = x + y + z$ and $v = x^2 + y^2 + z^2$

Then the given equation is

$$f(u, v) = 0 \quad \dots(2)$$

Diff. (2) partially w.r.t x , we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0$$

$$\frac{\partial f}{\partial u}(1+p) + \frac{\partial f}{\partial v}(2x+2zp) = 0 \quad \dots(3)$$

diff. (2) Partially w.r.t y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial f}{\partial u}(1+q) + \frac{\partial f}{\partial v}(2y+2zq) \quad \dots(4)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between (3) and (4), we get

$$\frac{1+p}{1+q} = \frac{x+zp}{y+zq}$$

or $(y-z)p + (z-x)q = x-y$

Example 7 : From the partial differential equation from

$$z = f(x^2 - y^2)$$

Solution : $z = f(x^2 - y^2)$

Differentiating (1) w.r.t. x and y

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)2x \quad \dots(1)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \quad \dots(2)$$

Dividing (2) by (3) we get $\frac{p}{q} = \frac{-x}{y}$ or $py = -qx$

or $yp + xq = 0$ Ans.

Example 8 : $(x-h)^2 + (y-k)^2 + z^2 = a^2$

Solution : We have

$$(x-h)^2 + (y-k)^2 + z^2 = a^2 \quad \dots(1)$$

Differentiating equation (1) partially w.r.t x and y, we get

$$x-h+zp=0$$

and $y-k+zq=0$

Putting the values of $x-h$ and $y-k$ from the last two equations in the given equation (1), we get

$$z^2(p^2 + q^2 + 1) = a^2$$

Which is the required P.D.E.

Example 9 : $z = (x+a)(y+b)$

Solution : We have

$$z = (x+a)(y+b) \quad \dots(1)$$

Differentiating equation (1) partially w.r.t x and y we get

$$\frac{\partial z}{\partial x} = p = (y+b) \quad \dots(2)$$

and $\frac{\partial z}{\partial y} = q = (x+a) \quad \dots(3)$

Putting the values, of $(y+b)$ and $(x+a)$ from (2) and (3), we get in the equation (1), we get

$$z = pq, \text{ which is the required P.D.E.}$$

Example 10 : $z = f(x^2 + y^2)$

Solution : We have

$$z = f(x^2 + y^2) \quad \dots(1)$$

Diff. (1) partially w.r.t x and y, we get

$$p = f'(x^2 + y^2) \cdot 2x \quad \dots(2)$$

$$q = f'(x^2 + y^2) \cdot 2y \quad \dots(3)$$

From (2) and (3), we get

$py - qx = 0$, which is the required P.D.E.

Example 11 : $f(x + y + z, x^2 + y^2 + z^2) = 0$

Solution : We have,

$$f(x + y + z, x^2 + y^2 + z^2) = 0 \quad \dots(1)$$

Let, $u = x + y + z$ and $v = x^2 + y^2 + z^2$

Then the given equation is

$$f(u, v) = 0 \quad \dots(2)$$

Diff. (2) partially w.r.t x, we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0$$

$$\frac{\partial f}{\partial u} (1 + p) + \frac{\partial f}{\partial v} (2x + 2zp) = 0 \quad \dots(3)$$

diff. (2) Partially w.r.t y, we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) = 0$$

$$\frac{\partial f}{\partial u} (1 + q) + \frac{\partial f}{\partial v} (2y + 2zq) = 0 \quad \dots(4)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between (3) and (4), we get

$$\frac{1 + p}{1 + q} = \frac{x + zp}{y + zq}$$

$$\text{or } (y - z)p + (z - x)q = x - y$$

EXERCISE 1.1 :

From the differential equation

$$z = (x + a)(y + b)$$

$$(x - h)^2 + (y - k)^2 + z^2 = a^2$$

$$(3) \quad 2z = (ax + y)^2 + b$$

$$(4) \quad ax^2 + by^2 + z^2 = 1$$

$$(6) \quad x^2 + y^2 = (z - c)^2 \tan^2 \alpha$$

$$(8) \quad z = f(x^2 + y^2)$$

$$(7) \quad 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

ANSWER 1.1 :

$$(1) \quad pq = 1$$

$$(2) \quad z^2(p^2 + q^2 + 1) = e^2$$

$$(3) \quad px + qy = q^2$$

$$(4) \quad z(px + qy) = z^2 - 1$$

$$(5) \quad yp - xq = 0$$

$$(6) \quad yp - xq = 0$$

$$(7) \quad 2z = xp + qy$$

1.8 SOLUTION OF EQUATION BY DIRECT INTEGRATION :

Example 12 : Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Solution : $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Integrating w.r.t. 'x' we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2} \sin(2x+3y) + f(y)$$

Integrating w.r.t x, we get

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{1}{4} \cos(2x+3y) + x[f(y)dy + g(y)] \\ &= -\frac{1}{4} \cos(2x+3y) + x\phi(y) + g(y) \end{aligned}$$

Integrating w.r.t. 'y', we get

$$z = -\frac{1}{2} \sin(2x+3y) + x \int \phi(y) dy + \int g(y) dy$$

$$z = -\frac{1}{2} \sin(2x+3y) + x\phi_1(y) + \phi_2(y) \quad \text{Ans}$$

Example 13 : Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Subject to the condition $z(x,0) = x^2$ and $z(1,y) = \cos y$

Solution : $\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = x^2 y$

On integrating w.r.t x, we obtain

$$\frac{\partial z}{\partial y} = \frac{x^3}{3} y + f(y)$$

Integrating w.r.t y, we obtain

$$z = \frac{x^3}{3} \cdot \frac{y^2}{2} + \int f(y) dy + g(x)$$

$$[F(y) = \int f(y) dy]$$

$$z = \frac{x^3}{6} y^2 + f(y) + g(x) \quad (1)$$

Condition 1 : Putting $z = x^2$ and $y = 0$ in (1), we get

$$x^2 = 0 + F(0) + g(x)$$

Putting the value of $g(x)$ in (1), we get

$$z = \frac{x^3 y^2}{6} + F(y) + x^2 - F(0) \quad (2)$$

Condition 2 : $z(1,y) = \cos y$

Putting $x = 1$ and $z = \cos y$ in (2), we get

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

Putting the value of $F(y)$ in (2), we obtain

$$z = \frac{1}{6} x^3 y^2 + \cos y - \frac{1}{6} y^2 - 1 + x^2 \quad \text{Ans.}$$

Example 14 : Solve $\frac{\partial^2 z}{\partial y^2} = z$, if $y = 0, z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

Solution : If z is a function of y alone, then

$$z = A \sinh y + B \cosh y \text{ where } A \text{ and } B$$

and constants, since z is a function of x and y , A and B are the function of x .

$$\text{Then} \quad z = \sinh y \cdot f(x) + \cosh y \cdot \phi(x) \quad \dots(1)$$

On putting $y = 0$ and $z = e^x$ in (1), we get

$$e^x = \phi(x)$$

$$(1) \text{ becomes } z = \sinh y \cdot f(x) + \cosh y \cdot e^x \quad \dots(2)$$

On differentiating (2) w.r.t y , we get

$$\frac{\partial z}{\partial y} = \cosh y \cdot f(x) + \sinh y \cdot e^x \quad \dots(3)$$

On putting $y = 0$ and $\frac{\partial z}{\partial y} = e^{-x}$ in (3), we obtain

$$e^{-x} = f(x)$$

$$(2) \text{ becomes, } z = e^{-x} \sinh y + e^x \cosh y$$

Ans.

Example 15 : Solve $\frac{\partial^2 z}{\partial x \partial y} = xy^2$

Solution : $\frac{\partial^2 z}{\partial x \partial y} = xy^2$

On integrating w.r.t x we get

$$\frac{\partial z}{\partial y} = \frac{x^2}{2} y^2 + f(y)$$

Integrating w.r.t y we get

$$z = \frac{x^2}{2} \frac{y^3}{3} + f(y) + \phi(x)$$

$$z = \frac{x^2 y^3}{6} + f(y) + \phi(x)$$

Example 16 : Solve $\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$

Solution : Integrating w.r.t x we get

$$\frac{\partial z}{\partial y} = e^y \sin x + f(y)$$

Integrating w.r.t y we get

$$z = \sin x e^y + f(y) + \phi(x)$$

EXERCISE 1.2 :

Solve the following.

$$(1) \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$$

$$(2) \quad \frac{\partial^2 z}{\partial x^2} = a^2 z, \text{ when } x=0, \frac{\partial z}{\partial x} = a \sin y \text{ and } \frac{\partial z}{\partial y} = 0$$

$$(3) \quad \frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \text{ if } \frac{\partial z}{\partial y} = -2 \sin y \text{ and } z = 0 \text{ when } x = 0$$

$$(4) \quad \text{The partial differential equation } y \frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = 0 \text{ is elliptic if}$$

$$(a) x^2 = y^2 \quad (b) x^2 < y^2 \quad (c) x^2 + y^2 > 1 \quad (d) x^2 + y^2 = 1$$

ANSWER 1.2 :

$$(1) \quad z = \frac{y^2}{2} \log x + 2xy + f(y) + \phi(x)$$

$$(2) \quad z = \sin x + e^y \cos x$$

LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE

$$Pp + Qq = R, \text{ where, } P, Q, R \text{ are the function of } x, y, z \text{ and } P = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$$

$$\text{Solution:- } Pp + Qq = R \quad \dots(1)$$

This form of the equation is obtained by eliminating an arbitrary function f from

$$f(u, v) = 0 \quad \dots(2)$$

where u, v are function x, y, z

Differentiating (2) partially w.r.t to x and y

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0 \quad \dots(3)$$

$$\text{and } \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0 \quad \dots(4)$$

Let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4)

$$\text{From (3)} \quad \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) = - \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) \quad \dots(5)$$

$$\text{From (4)} \quad \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} q \right) = - \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} q \right) \quad \dots(6)$$

Dividing (5) by (6), we get

$$\frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p}{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q} = \frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p}{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q}$$

$$\text{or } \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right] \left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right] = \left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right] \left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right]$$

$$\frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial z} q + \frac{\partial u}{\partial z} p \times \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} p \times \frac{\partial v}{\partial z} p q$$

$$\frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} p + \frac{\partial u}{\partial z} q \times \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} q \times \frac{\partial v}{\partial z} p q$$

$$= \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} p + \frac{\partial u}{\partial z} q \times \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} q \times \frac{\partial v}{\partial z} p q$$

$$\text{or } \left[\frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial y} \right] p + \left[\frac{\partial u}{\partial z} \times \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial z} \right] q$$

$$= \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x} \quad \dots(7)$$

If (1) and (7) are the same, then the coefficient p, q are equal

$$P = \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial z} \quad \dots(8)$$

$$R = \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x}$$

Now suppose $u = c_1$ and $v = c_2$ are two solutions, where a, b are constants.

Differentiating $u = c_1$ and $v = c_2$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \dots(9)$$

$$\text{and } \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \quad \dots(10)$$

Solving (9) and (10), we get

$$\frac{dx}{\frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \times \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x}} \quad \dots(11)$$

From (8) and (11)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Solution of these equations are

$$u = c_1 \quad \text{and} \quad v = c_2$$

$\therefore f(u, v) = 0$ is the required solution of (1).

*WORKING RULE :

First step : Write down the auxillary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$Pp + Qq = R$$

Second step. Solve the above auxillary equations

Let the two solutions be $u = c_1$ and $v = c_2$

Third step. Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$$Pp + Qq = R$$

*Example 17 : Solve the following partial differential equaion

$$yq - xp = z,$$

$$\text{where } p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$$

Solution : $yq - xp = z$

Here the auxillary equation are

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

$$-\log x = \log y - \log a$$

$$xy = a$$

$$\log y = \log z + \log b$$

(From first two equations)

...(1)

(From last two equations)

$$\frac{y}{z} = b \quad \dots(2)$$

From (1) and (2)

Hence the solution is $f\left(xy, \frac{y}{z}\right) = 0$ Ans.

*Example 18 : Solve $y^2 p - xyq = x(z - 2y)$

olution : $y^2 p - xyq = x(z - 2y)$

The auxillary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \quad \dots(1)$$

Considering first two members of the equations

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad x dx = -y dy$$

$$\text{Integrating} \quad \frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2}$$

$$\text{or} \quad x^2 + y^2 = C_1 \quad \dots(2)$$

From last two equations of (1)

$$-\frac{dy}{y} = \frac{dz}{z - 2y}$$

$$\text{or} \quad -z dy + 2y dy = y dz \quad \text{or} \quad 2y dy = y dz + z dy$$

$$y^2 = yz + C_2 \quad \dots(3)$$

$$y^2 = yz + C_2$$

From (2) and (3)

$$x^2 + y^2 = f(y^2 - yz) \quad \text{Ans}$$

Example 19 : Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution : $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

The auxillary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(x + y + z)(y - z)} = \frac{dz - dx}{(x + y + z)(z - x)}$$

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

Integrating first two members of (2), we have

$$\log(x - y) = \log(y - z) + \log c_1$$

$$\log \frac{x - y}{y - z} = \log C_1 \quad \text{or} \quad \frac{x - y}{y - z} = C_1$$

Similarly from last two members of (2), we have

$$\frac{y - z}{z - x} = c_2$$

The required solution is

$$f\left[\frac{x - y}{y - z}, \frac{y - z}{z - x}\right] = 0$$

METHOD OF MULTIPLIERS :-

Let the auxillary equations be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

l, m, n may be contains or functions of x, y, z then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

l, m, n are chosen in such a way that

$$lP + mQ + nR = 0$$

Thus $l dx + m dy + n dz = 0$

Solve the differential equation, if the solution is $u = c_1$

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is $v = c_2$.

$\therefore$ Required solution $f(u, v) = 0$

Example 20 : Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$

Solution : $(mz - ny)\frac{\partial x}{\partial x} + (nx - ly)\frac{\partial z}{\partial y} = ly - mx$

Here, the auxillary equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - ly} = \frac{dz}{ly - mx}$$

Using multipliers x, y, z we get

each fraction

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - nx)} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

which on integration gives

$$x^2 + y^2 + z^2 = c_1 \quad \dots(1)$$

Again using multipliers, l, m, n , we get

each fraction

$$= \frac{lxdx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{lxdx + mdy + ndz}{0}$$

$$\therefore lxdx + mdy + ndz = 0$$

which, on integration gives

$$lx + my + nz = c_2 \quad \dots(2)$$

Hence from (1) and (2), the required solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz) \quad \text{Ans.}$$

Example 21 : Find the genreal solution of $x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2)$

$$\text{Solution : } x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2) \quad \dots(1)$$

The auxillary simulatneous equtaions are

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \quad \dots(2)$$

Using multipliers x, y, z we get

Each term of (2) is equal to

$$\frac{xdx + ydy + zdz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

$$\text{On integration } x^2 + y^2 + z^2 = C_1 \quad \dots(3)$$

Again (2) can be written as

$$\frac{\frac{dx}{x}}{z^2 - y^2} = \frac{\frac{dy}{y}}{x^2 - z^2} = \frac{\frac{dz}{z}}{y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)} \quad \dots(4)$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\text{or } \log x + \log y + \log z = \log C_2$$

$$\text{or } \log xyz = \log C_2 \quad \text{or } xyz = C_2 \quad \dots(5)$$

From (3) and (5), the general solution is

$$xyz = f(x^2 + y^2 + z^2) \quad \text{Ans.}$$

Example 22 : Solve the partial differential equation

$$\frac{y-z}{yz}p = \frac{z-x}{zx}q = \frac{x-y}{xy}r$$

$$\text{Solution : } \frac{y-z}{yz}p = \frac{z-x}{zx}q = \frac{x-y}{xy}r$$

Multiplying by xyz , we get

$$x(y-z)p + y(z-x)q = z(x-y)r$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$\frac{dx+dy+dz}{0} \quad \dots(1)$$

$$dx+dy+dz=0$$

Which on integration gives

$$x+y+z=0$$

Again(1) can be written

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(y-z)+(z-x)+(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad \dots(2)$$

On integration we get

$$\log x + \log y + \log z = \log b \quad \text{or} \quad \log xyz = \log b \quad \text{or} \quad xyz = b \quad \dots(3)$$

From (2) and (3) the general solution is

$$xyz = f(x+y+z) \quad \text{Ans.}$$

*** Example 23 :** Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Solution : $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Here the auxillary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

From the last two members of (2) we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Which on integration gives

$$\log y = \log z + \log a \quad \text{or} \quad \log \frac{y}{z} = \log a$$

$$\frac{y}{z} = a \quad \dots(3)$$

Using multipliers x, y, z we have

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)} = \frac{dz}{z}$$

Which on integration gives

$$\log(x^2 + y^2 + z^2) = \log z + \log b$$

$$\frac{x^2 + y^2 + z^2}{z} = b \quad \dots(4)$$

Hence from (3) and (4), the required solution is

$$x^2 + y^2 + z^2 = 2f\left(\frac{y}{z}\right) \quad \text{Ans.}$$

*** Example 24 :** Solve the differential equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$

$$\text{olution : } x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z \quad \dots(1)$$

The auxillary equations of (1) are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad \dots(2)$$

Take first two members of (2) and integrate then

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$\frac{1}{x} - \frac{1}{y} = c_1$$

(2) can be written as

$$\frac{\frac{dx}{x}}{\frac{y}{x+y}} = \frac{\frac{dy}{y}}{\frac{y}{x+y}} = \frac{\frac{dz}{z}}{\frac{y}{x+y}} = \frac{\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z}}{(x+y) - (x+y)}$$

or $\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$

On integration we get

or $\log x + \log y - \log z = \log c_2$

or $\log \frac{xy}{z} = \log c_2$ or $\frac{xy}{z} = c_2$

From (3) and (4) we have

$$f\left[\frac{1}{x} - \frac{1}{y}, \frac{xy}{z}\right] = 0 \quad \text{Ans.}$$

Example 25 : Find the general solution of

$$x \frac{\partial x}{\partial x} + y \frac{\partial z}{\partial y} + t \frac{\partial z}{\partial t} = xyt$$

Solution : The auxillary equation are

$$\frac{dx}{x} = \frac{dy}{y} + \frac{dt}{t} = \frac{dz}{xyt} \quad \dots(1)$$

Taking the first members and integrating, we get

$$\log x = \log y + \log t = \log at$$

or $x = at, \text{ i.e. } y/x = a$

Similarly, from the 2nd and 3rd members

$$\frac{t}{y} = b$$

Multiplying the equations (1) by xyt , we get

$$dz = \frac{tydx}{1} = \frac{txdy}{1} = \frac{xydt}{t} = \frac{tydx + txdy + xydt}{3}$$

Integrating

$$z = \frac{1}{3}xyt + c \quad \text{or} \quad z - \frac{1}{3}xyt = c$$

From (2), (3) and (4) the solution is

$$z - \frac{1}{3}xyt = f\left(\frac{y}{x}\right) + \phi\left(\frac{t}{y}\right) \quad \text{Ans.}$$

Example 26 : Solve $(y+z)p - (x+z)q = x-y$

Solution : $(y+z)p - (x+z)q = x-y$

The auxillary equations are

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$$

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y+z-(x+z)+x-y}$$

$$\frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

Thus we have

$$dx + dy + dz = 0$$

which on integration gives

$$x + y + z = c_1$$

Let us use multipliers $(x, y, -z)$ for (2)

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{xdx + ydy - zdz}{x(y+z) - y(x+z) - z(x-y)}$$

or

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{xdx + ydy - zdz}{0}$$

Integrating $xdx + ydy - zdz = 0$, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

or

$$x^2 + y^2 - z^2 = 2c_2$$

From (3) and (4), we get the required solution

$$f(x + y + z, x^2 + y^2 - z^2) = 0 \quad \text{Ans.}$$

Example 27 : Solve $zp + yq = x$

Solution : $zp + yq = x$

The auxillary equations are

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

$$(i) \quad (ii) \quad (iii)$$

From (i) and (ii)

$$\frac{dx}{z} = \frac{dz}{x}$$

or

$$x dx = z dz$$

$$\text{or} \quad \frac{x^2}{2} = \frac{z^2}{2} - \frac{c_1}{2} \quad \text{or} \quad x^2 = z^2 - c_1 \quad \dots(2)$$

$$\text{or} \quad z = \sqrt{x^2 + c_1}$$

Putting the value of z in (i),

$$\frac{dx}{\sqrt{x^2 + c_1}} = \frac{dy}{y}$$

$$\sinh^{-1} \frac{x}{\sqrt{c_1}} = \log y + c_2 \quad \text{or} \quad \sinh^{-1} \frac{x}{\sqrt{c_1}} - \log y = c_2 \quad \dots(3)$$

From (2) and (3), the required solution is

$$f(z^2 - x^2) = \sinh^{-1} \frac{x}{\sqrt{c_1}} - \log y \quad \text{Ans.}$$

Example 28 : Solve $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

Solution : $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

$$\text{or} \quad px(z - 2y^2) + qy(z - y^2 - 2x^3) = z(z - y^2 - 2x^3) \quad \dots(1)$$

Here the auxillary equation are

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)} \quad \dots(2)$$

From the last two members of (2) we have

$$\frac{dy}{y} = \frac{dz}{z}$$

Which gives on integration

$$\log y = \log z + \log a \quad \text{or} \quad y = az \quad \dots(3)$$

From the first and third members of (2) we have

$$\frac{dx}{x(2-2y^2)} = \frac{dz}{z(z-y^2-2x^3)} \quad \text{put } y = az$$

$$\text{or } \frac{dx}{x(z-2a^2z^2)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

$$\frac{dx}{x(x-2a^2z)} = \frac{dz}{z-a^2z^2-2x^3}$$

$$\text{or } zdx - a^2z^2dx - 2x^3dx = xdz - 2a^2xzdz$$

$$\text{or } (xdz - zdx) - a^2(2xzdx - z^2dx) + 2x^3dx = 0$$

$$\text{or } \frac{xdz - zdx}{x^2} - a^2 \frac{(2xzdx - z^2dx)}{x^2} + 2x^3dx = 0$$

On integrating, we have

$$\frac{z}{x} - a^2 \frac{z^2}{x} + x^2 = b$$

From (3) and (4), we have

$$\frac{y}{z} = f\left(\frac{z}{x} - \frac{a^2z^2}{x} + x^2\right)$$

Ans

Example 29 : Solve $p \tan x + q \tan y = \tan z$

Solution : We have $p \tan x + q \tan y = \tan z$

The Lagrange's auxillary equations of the given differential equation (1) are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two members, we get

$$\cot x dx = \cot y dy$$

Integrating,

$$\log \sin x = \log \sin y + \log C_1$$

$$\text{Or } \frac{\sin x}{\sin y} = C_1$$

Again taking the last two members, we get

$$\cot y dy = \cot z dz$$

Integrating,

$$\log \sin y = \log \sin z + \log C_2$$

$$\text{or } \frac{\sin y}{\sin z} = C_2$$

Hence the general solution of (1) is given by

$$\frac{\sin y}{\sin z} = f\left(\frac{\sin x}{\sin y}\right)$$

$$\text{or } f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

Example 30 : Solve $pz - qz = z^2 + (x+y)^2$

Solution : We have

$$pz - qz = z^2 + (x+y)^2 \quad \dots(1)$$

The Lagrange's auxillary equations of (1) are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

Taking the first two members, we get

$$dx + dy = 0$$

Integrating, $x + y = C_1$

Again taking the first and the last members, we get

$$\frac{z dz}{z^2 + (x + y)^2} = dx$$

or $\frac{dz}{z^2 + C_1^2} = 2 dx$

Integrating, $\log(x^2 + C_1^2) + 2x = C_2$

or $\log(z^2 + x^2 + y^2 + 2xy) - 2x = C_2$

Hence the general solution of (1) is given by

$$\log(x^2 + y^2 + z^2 + 2xy) - zx = f(x + y)$$

where f is an arbitrary function

or $[z^2 + (x + y)^2]e^{-zx} = f(x + y)$

Example 31 : Solve $x^2 p + y^2 q = z^2$

Solution : We have $x^2 p + y^2 q = z^2$

The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

From the first two members, we get

$$\frac{1}{x} - \frac{1}{y} = C_1$$

From the last two members, we get

$$\frac{1}{y} - \frac{1}{z} = C_2$$

Hence the general solution of (1) is given by

$$\frac{1}{y} - \frac{1}{z} = f\left(\frac{1}{x} - \frac{1}{y}\right)$$

or $f\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{y} + \frac{1}{z}\right) = 0$

Example 32 : Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

Solution : We have

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx \quad \dots(1)$$

The Lagrange's auxiliary equations of the given differential equation (1) are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

Taking the last two members, we get

$$(y - z)dy = (y + z)dz$$

$$dy - (zdy + ydz) - z dz = 0$$

Integrating, $y^2 - 2yz - z^2 = C_1$

Again choosing x, y, z as multipliers, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating $x^2 + y^2 + z^2 = C_2$

Hence the general solution of (1) is given by

$$f(y^2 - 2yz - z^2, x^2 + y^2 + z^2) = 0$$

or $y^2 - 2yz - z^2 = \phi(x^2 + y^2 + z^2)$

EXERCISE 1.3 :

Solve the following partial differential equations :

(1) $y \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z^2 + 1$

(2) $(y-z)p + (x-y)q = z-x$

(3) $(y+zx)p - x(x+yz)q = x^2 - y^2$

(4) $zx \frac{\partial z}{\partial x} - zy \frac{\partial z}{\partial y} = y^2 - x^2$

(5) $p+q+2xz=0$

(6) $(x^2+y^2)p + 2xyq = (x+y)^z$

(7) $\frac{\partial z}{\partial x} = 2 \frac{\partial z}{\partial y} = 2x - e^y + 1$

(8) $p+3q=5z+\tan(y-3x)$

(9) $xp - yq + x^2 - y^2 = 0$

(10) $(x+y) \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \right) = z-1$

(11) $3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0, u(x,0) = 4e^x$

(12) $4 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \text{ when } t=0, u=3e^{-x} - e^{-5x}$

(13) Find the solution of the equation $\frac{x \partial z}{\partial y} - \frac{y \partial z}{\partial x} = 0$, which passes through the curve $z=1, x^2+y^2=4$

Answer 1.3:

(1) $f(x-y) = \log y - \tan^{-1} z$

(2) $f(x+u+z, x^2+2yz)$

(3) $f(x^2+y^2-z^2) = (x-y)^2 - (z+1)^2$

(4) $f(x^2+y^2+z^2, xy) = 0$

(5) $f(x-y) = x^2 + \log z$

(6) $f\left(\frac{x+y}{z}, \frac{2y}{x^2-y^2}\right) = 0$

(7) $f(2x+y) = z - \frac{(2x+1)^2}{4} - \frac{e^y}{2}$

(8) $f(y-3x) = \frac{e^{5x}}{5z + \tan(y-3x)}$

(9) $f(xy) = \frac{x^2}{2} + \frac{y^2}{2} + z$

(10) $f(x-y) = \frac{x+y}{(z-1)^2}$

(11) $u = ue^{-x+\frac{3y}{2}}$

(12) $u = 3e^{-x+t} - 3e^{-5x+2t}$

(13) $f(x^2+y^2-4, z=1) = 0$

MULTIPLE CHOICE QUESTIONS

 01. The partial differential equation from $z = ax + by + a^2 + b^2$ is

(a) $z = px + qy + p^2 + q^2$

(b) $z = px + qy + p - q$

(c) $z = px - qy + p^2 - q^2$

(d) None of these

 02. The partial differential equation from $z = f_1(x) \cdot f_2(y)$ is...

(a) $z \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$

(b) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$

(c) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y}$

(d) None

 03. The partial differential equation from $z = f(x^2 - y^2)$ is...

(a) $py + qx = 0$

(b) $py - qx = 0$

(c) $qy - px = 0$

(d) $qy + px = 0$

 04. Order of the equation $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$ is

(a) 1st order

(b) 2nd order

(c) 3rd order

(d) 4th order

05. The P.D.E. from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is,...

☒ (a) $2z = xp + yq$

(c) $z = xp + yq$

(b) $2z = xp - yq$

(d) $z = xp - yq$

06. Solution of $\frac{\partial^2 z}{\partial x^2} = xy$ is,...

(a) $z = \frac{1}{2}x^3y + xf(y) + g(y)$

(c) $z = \frac{1}{6}x^2y^2 + xf(y) + g(y)$

☒ (b) $z = \frac{1}{6}x^3y + xf(y) + g(y)$

(d) $z = \frac{1}{6}xy^2 + xf(y) + g(y)$

07. Order of the equation $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ is,...

(a) 1st order

☒ (c) 3rd order

(b) 2nd order

(d) None of these

08. Choose the correct linear P.D.E.

(a) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$

(c) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial^3 z}{\partial y^3}\right) = 2x\left(\frac{\partial z}{\partial x}\right)$

☒ (b) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

(d) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$

09. Degree of P.D.E $y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right)$ is

(a) One

(c) Third

☒ (b) Two

(d) Zero

10. The equation $p \tan y + q \tan x = \sec^2 z$ is of order

☒ (a) 1

(3) 0

(b) 2

(d) None of these

11. Equation $\frac{\partial^2 z}{\partial x^2} = 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial y}\right) = 0$ is of order

(a) 1

(c) 3

(b) 2

(d) None of these

12. The equation $(2x+3y)p + 4xq - 8pq = x + y$ is,...

(a) linear

(c) quasi-linear

☒ (b) non-linear

(d) semi linear

13. Partial differential equation obtained by eliminating arbitrary constants a & b from $z = ax^2 - by^2$ is

(a) $z = px + qy$

(c) $2z = px + qy$

☒ (b) $2z = px - qy$

(d) $z = ap^2 - bq^2$

14. The partial differential equation obtained by eliminating the arbitrary function ϕ from $z = \phi(x+y)$ is

(a) $z = pq$

(c) $pq = 1$

(b) $z = p + q$

☒ (d) $p = q$

15. The equation $P_p + Q_q = R$ is known as

(a) Charpits equation

(c) Bernoullis equation

☒ (b) Lagrange's equation

(d) Clairant's equation

16. The Lagrange's auxiliary equations for the partial differential equation $P_p + Q_q = R$ are

☒ (a) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(c) $\frac{dx}{P} = \frac{dz}{R}$

(b) $\frac{dx}{P} = \frac{dy}{Q}$

(d) None

17. The general solution of the linear partial differential equation $P_p + Q_q = R$ is

(a) $f(u, v) = 1$

(b) $f(u, v) = -1$

- (c) $f(u, v) = 0$ (d) None
18. Solution of partial differential equation $p + q + 2xz = 0$ is
 (a) $f(x - y) = x^2 + \log z$ (b) $f(x + y) = x + \log z$
 (c) $f(x - y) = x + \log z$ (d) $f(x - y) = y + \log x$
19. Solution of $\frac{\partial^2 z}{\partial x^2} + z = 0$, when $x = 0, z = e^y$ & $\frac{dz}{dx} = 1$ is,..
 (a) $z = \sin x + \cos x$ (b) $z = \sin x + e^y \cos x$
 (c) $z = \cos x + e^y \sin x$ (d) None
20. The solution of $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ is,..
 (a) $\log(x^2 + y^2 + z^2)$ (b) $\frac{x^2 + y^2 + z^2}{z}$
 (c) both a and b (d) None of these
21. A linear partial differential equation of first order, commonly known as
 (a) Charpit's equation (b) Laplace equation
 (c) Lagrange's linear equation (d) Bernoulli's equation

ANSWER KEY (MULTIPLE CHOICE QUESTIONS)

- | | | | | | |
|--------|--------|---------|---------|---------|---------|
| 01.(a) | 02.(a) | 03. (a) | 04. (b) | 05. (a) | 06. (b) |
| 07.(c) | 08.(b) | 09. (b) | 10. (a) | 11. (b) | 12. (b) |
| 13.(b) | 14.(d) | 15. (b) | 16. (a) | 17. (c) | 18. (a) |
| 19.(b) | 20.(c) | 21. (c) | | | |

2

PARTIAL DIFFERENTIAL EQUATIONS
NON-LINEAR IN p AND q

2.1 INTRODUCTION :

We give below the methods of solving non-linear partial differential equations in certain standard form only.

Type 1 : Equation of the type $f(p, q) = 0$, i.e. equations containing p and q only.

Method : Let the required solution be

$$z = ax + by + c \quad \dots(1)$$

$$\frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b$$

On putting these values in $f(p, q) = 0$

we get $f(a, b) = 0$

From this, find the value of b in terms of a and substitute the value of b in (1), that will be the required solution.

Example 1 : Solve $p^2 + q^2 = 1$

Solution : Let $z = ax + by + c$

$$\therefore p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

On substituting the value of p and q in (1), we have

$$\therefore a^2 + b^2 = 1 \quad \text{or} \quad b = \sqrt{1 - a^2}$$

- (c) $f(u, v) = 0$ (d) None
18. Solution of partial differential equation $p + q + 2xz = 0$ is
 (a) $f(x - y) = x^2 + \log z$ (b) $f(x + y) = x + \log z$
 (c) $f(x - y) = x + \log z$ (d) $f(x - y) = y + \log x$
19. Solution of $\frac{\partial^2 z}{\partial x^2} + z = 0$, when $x = 0, z = e^y$ & $\frac{dz}{dx} = 1$ is,..
 (a) $z = \sin x + \cos x$ (b) $z = \sin x + e^y \cos x$
 (c) $z = \cos x + e^y \sin x$ (d) None
20. The solution of $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ is,..
 (a) $\log(x^2 + y^2 + z^2)$ (b) $\frac{x^2 + y^2 + z^2}{z}$
 (c) both a and b (d) None of these
21. A linear partial differential equation of first order, commonly known as
 (a) Charpit's equation (b) Laplace equation
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ANSWER KEY (MULTIPLE CHOICE QUESTIONS)

- | | | | | | |
|--------|--------|--------|--------|--------|--------|
| 01.(a) | 02.(a) | 03.(a) | 04.(b) | 05.(a) | 06.(b) |
| 07.(c) | 08.(b) | 09.(b) | 10.(a) | 11.(b) | 12.(b) |
| 13.(b) | 14.(d) | 15.(b) | 16.(a) | 17.(c) | 18.(a) |
| 19.(b) | 20.(c) | 21.(c) | | | |

2

PARTIAL DIFFERENTIAL EQUATIONS NON-LINEAR IN p AND q

2.1 INTRODUCTION :

We give below the methods of solving non-linear partial differential equations in certain standard form only.

Type 1 : Equation of the type $f(p, q) = 0$, i.e. equations containing p and q only.

Method : Let the required solution be

$$z = ax + by + c \quad \dots(1)$$

$$\frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b$$

On putting these values in $f(p, q) = 0$

we get $f(a, b) = 0$

From this, find the value of b in terms of a and substitute the value of b in (1), that will be the required solution.

Example 1 : Solve $p^2 + q^2 = 1$

Solution : Let $z = ax + by + c$

$$\therefore p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

On substituting the value of p and q in (1), we have

$$\therefore a^2 + b^2 = 1 \quad \text{or} \quad b = \sqrt{1 - a^2}$$

(5) $z = ax + \sqrt{(4^2 - a^2)}y + c$

(6) $z = ax - \frac{a}{a-1}y + c$

Type II : Equation of the type

$$z = px + qy + f(p, q)$$

Its solution is $z = ax + by + f(a, b)$

Example 4 : Solve $z = px + qy + p^2 + q^2$

Solution : $z = px + qy + p^2 + q^2$ $p = a, q = b$

Its solution is $p = a, q = b \Rightarrow z = ax + by + a^2 + b^2$ Ans.

Example 5 : Solve $z = px + qy + 2\sqrt{pq}$

Solution : $z = px + qy + 2\sqrt{pq}$

Its solution is $z = ax + by + 2\sqrt{ab}$ Ans.

TYPE III : Equation of the type $f(z, p, q) = 0$ i.e. equations not containing x and y.

Let z be function of u where

$$u = x + ay$$

$$\frac{\partial u}{\partial x} = 1 \text{ and } \frac{\partial u}{\partial y} = a$$

Then $p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}(a)$$

On putting the values of p and q in the given equation $f(z, p, q) = 0$, it becomes $f\left(\frac{dz}{du}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the first order.

Rule. Assume $u = x + ay$ replace p and q by $\frac{dz}{dy}$ and $a\frac{dz}{du}$ in the given equation and then solve the ordinary differential equation obtained.

Example 6 : Solve $p(1+q) = qz$

Solution : $p(1+q) = qz$... (1)

$$u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = \frac{dz}{du} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = \frac{dz}{du}(a) \text{ (1) becomes}$$

$$\frac{dz}{du} \left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z \text{ or } 1 + a\frac{dz}{du} = az$$

$$\text{or } a\frac{dz}{du} = az - 1 \text{ or } \frac{dz}{du} = \frac{az-1}{a} \text{ or } \frac{du}{dz} = \frac{a}{az-1} \text{ or } du = \frac{a dz}{az-1}$$

$$u = \log(az-1) + \log c$$

$$x + ay = \log c(az-1) \text{ Ans.}$$

Example 7 : Solve $p(1+q^2) = q(z-a)$

Solution : Let $u = x + by$

So that $p = \frac{dz}{du}$ and $q = b\frac{dz}{du}$

Substituting these values of p and q in the given equation we have

$$\frac{dz}{du} \left[1 + b^2 \left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a)$$

$$1 + b^2 \left(\frac{dz}{du} \right)^2 = b(z-a) \text{ or } b^2 \left(\frac{dz}{du} \right)^2 = bz - ab - 1$$

$$\frac{dz}{du} = \frac{1}{b} \sqrt{bz - ab - 1}$$

$$\int \frac{b dz}{\sqrt{bz - ab - 1}} = \int du + c$$

$$2\sqrt{bz - ab - 1} = u + c$$

$$4(bz - ab - 1) = (u + c)^2$$

$$4(bz - ab - 1) = (x + by + c)^2$$

Example 8 : Solve $z^2(p^2x^2 + q^2) = 1$

Ans.

Solution : $z^2(p^2x^2 + q^2) = 1$

...(1)

$$\text{or } z^2 \left[\left(x \frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1$$

$$\text{or } z^2 \left[\left(\frac{\frac{\partial z}{\partial x}}{x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1$$

$$\text{or } z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1$$

$$\text{where } \frac{\partial x}{x} = \partial X \text{ or } \log x = X$$

...(2)

Let $u = X + ay$

$$\frac{\partial z}{\partial X} = \frac{dz}{du} \text{ and } \frac{\partial z}{\partial y} = a \frac{dz}{du}$$

Then (2) becomes

$$z^2 \left[\left(\frac{dz}{du} \right)^2 + \left(a \frac{dz}{du} \right)^2 \right] = 1 \text{ or } \left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 = \frac{1}{z^2}$$

or

$$\left(\frac{dz}{du} \right)^2 = \frac{1}{z^2(1+a^2)} \text{ or } \frac{dz}{du} = \frac{1}{z\sqrt{1+a^2}} \text{ or } z dz = \frac{du}{\sqrt{1+a^2}}$$

or

$$\int z dz = \int \frac{du}{\sqrt{1+a^2}} + c \text{ or } \frac{z^2}{2} = \frac{u}{\sqrt{1+a^2}} + c$$

$$\sqrt{1+a^2} \frac{z^2}{2} = u + c\sqrt{1+a^2} = X + ay + c\sqrt{1+a^2}$$

$$= \log x + ay + c\sqrt{1+a^2}$$

Ans.

EXERCISE 2.2

1. $z^2(p^2 + q^2 + 1) = 1$

2. $x^2p^2 + y^2q^2 = z$

3. $1 + q^2 = q(z - a)$

ANSWER 2.2 :

(1) $(1 - z^2)^{\frac{1}{2}} = \frac{-(x+ay)}{\sqrt{1+a^2}} + c$

(2) $2\sqrt{z} = \frac{\log x + a \log y}{\sqrt{1+a^2}} + c$

(3) $\frac{x+by}{b} + \frac{1}{4}(z-a)^2 = \frac{1}{4}(z-a)\sqrt{(z-a)^2 - 2^2} + 4 \cosh^{-1} \left(\frac{z-a}{2} \right)$

***TYPE IV :** Equation of the type

$$f_1(x, p) = f_2(y, q)$$

In these equations, z is absent and the terms containing x and p can be written on one side and the terms containing y and q can be written on the other side.



Method : Let $f_1(x, p) = f_2(y, q) = a$

$f_1(x, p) = a$, solve it for p. Let $p = F_1(x)$

$f_2(y, q) = a$, solve it for q Let $q = F_2(y)$

Since $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ or $dz = p dx + q dy$

or $dz = F_1(x) dx + F_2(y) dy$ or $z = \int F_1(x) dx + \int F_2(y) dy + c$

Example 9 : Solve $p - x^2 = q + y^2$

Solution : $p - x^2 = q + y^2$ (say)

$p = x^2 + c$ and $q = c - y^2$

putting these values of p and q in

$dz = p dx + q dy = (x^2 + c) dx + (c - y^2) dy$

$z = \left(\frac{x^3}{3} + cx \right) + \left(cy - \frac{y^3}{3} \right) + c_1$

Ans.

Example 10 : Solve $p^2 + q^2 = z^2(x + y)$

Solution : $p^2 + q^2 = z^2(x + y)$ or $\left(\frac{p}{z} \right)^2 + \left(\frac{q}{z} \right)^2 = (x + y)$

or $\left(\frac{1}{z} \frac{\partial z}{\partial x} \right)^2 + \left(\frac{1}{z} \frac{\partial z}{\partial y} \right)^2 = x + y$ or $\left(\frac{\frac{\partial z}{\partial x}}{\frac{z}{\partial x}} \right)^2 + \left(\frac{\frac{\partial z}{\partial y}}{\frac{z}{\partial y}} \right)^2 = x + y$

or $\left(\frac{\partial Z}{\partial x} \right)^2 + \left(\frac{\partial Z}{\partial y} \right)^2 = x + y$

where $\frac{\partial Z}{z} = \partial Z$ or $\log z = Z$

or $p^2 + Q^2 = x + y$ or $P^2 - x = y - Q^2 = a$

$P^2 - x = a$ or $P = \sqrt{a + x}$

$y - Q^2 = a$ or $Q = \sqrt{y - a}$

Therefore, the equation $dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$

$dZ = P dx + Q dy$

gives

$dZ = \sqrt{a + x} dx + \sqrt{y - a} dy$

$Z = \int \sqrt{a + x} dx + \int \sqrt{y - a} dy + c$

or $\log z = \frac{2}{3}(a + x)^{\frac{3}{2}} + \frac{2}{3}(y - a)^{\frac{3}{2}} + c$ Ans.

EXERCISE 2.3 :

Solve

(1) $q - p + x - y = 0$

(2) $\sqrt{p} + \sqrt{q} = 2x$

(3) $q = xp + p^2$

(4) $z^2(p^2 + q^2) = x^2 + y^2$

(5) $z(p^2 - q^2) = x - y$

(6) $p^2 - q^2 = x - y$

(7) $(p^2 + q^2)y = qz$

ANSWER 2.3 :

(1) $2z = (x + a)^2 + (y + a)^2 + b$

(2) $z = \frac{1}{6}(2x - a)^3 + a^2y + b$

(3) $z = -\frac{x^2}{4} + \left\{ \frac{x\sqrt{x^2 + 4a}}{4} + a \log(x + \sqrt{x^2 + 4a}) \right\} + ay + b$

PARTIAL DIFFERENTIAL EQUATIONS NON-LINEAR IN p AND q / 54

$$(4) \quad z^2 = x\sqrt{x^2+a} + a \log(x + \sqrt{x^2+a}) + y\sqrt{y^2-a} - a \log(y + \sqrt{y^2-a}) + 2b$$

$$(5) \quad z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + b \quad (6) \quad z = \frac{2}{3}(x+c)^{3/2} + \frac{2}{3}(y+c)^{3/2} + c$$

$$(7) \quad z^2 = (cx+a^2) + c^2y^2$$

2.2 CHARPIT'S METHOD :

General method for solving partial differential equation with two independent variables

Solution : Let the general partial differential equation be

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

Since z depends on x, y we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \dots(2)$$

$$dz = p dx + q dy$$

The main thing in Charpits method is to find another relation between the variables x, y, z, p, q. Let the relation be

$$\phi(x, y, z, p, q) = 0$$

On solving (1) and (3), we get the values of p and q

These values of p and q when substituted in (2), it become integrable.

To determine ϕ , (1) and (3) are differentiated w.r.t x and y giving

$$\left. \begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} &= 0 \\ \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial x} &= 0 \end{aligned} \right\} \text{w.r.t } x \text{ (First pair)}$$

$$\left. \begin{aligned} \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial y} &= 0 \\ \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial y} &= 0 \end{aligned} \right\} \text{w.r.t } y \text{ (Second pair)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equation of first pair, we have

$$-\frac{\partial p}{\partial x} = \frac{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x}}{\frac{\partial f}{\partial p}} = \frac{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial x}}{\frac{\partial \phi}{\partial p}}$$

or

$$\left(\frac{\partial f}{\partial x} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial p} \right) + p \left(\frac{\partial f}{\partial z} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial p} \right) + \frac{\partial q}{\partial x} \left(\frac{\partial f}{\partial q} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q} \frac{\partial f}{\partial p} \right) = 0 \quad \dots(4)$$

On eliminating $\frac{\partial q}{\partial y}$ between the equations of second pair, we have

$$\left(\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial q} \right) + q \left(\frac{\partial f}{\partial z} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial q} \right) + \frac{\partial p}{\partial y} \left(\frac{\partial f}{\partial p} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p} \frac{\partial f}{\partial q} \right) = 0 \quad \dots(5)$$

Adding (4) and (5) and keeping in view the relation $\frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial q}{\partial x}$, the terms of the last brackets of (4) and (5) cancel. On rearranging, we get

$$\frac{\partial \phi}{\partial f} \left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) + \frac{\partial \phi}{\partial q} \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) + \frac{\partial \phi}{\partial z} \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) + \left(-\frac{\partial f}{\partial p} \right) \frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial y} = 0$$

$$\text{or} \quad \left(-\frac{\partial f}{\partial p} \right) \left(\frac{\partial \phi}{\partial z} \right) + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial y} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial z}$$

$$+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}=0 \quad \dots(6)$$

Equation (6) is Lagrange's linear equation of the first order with x, y, z, p, q as independent variables and ϕ as dependent variable. Its subsidiary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{d\phi}{0} \quad \dots(7)$$

Any of the integrals of (7) satisfies (6). Such an integral involving p or q or both may be taken as assumed relation (3). However, we should choose the simplest integral involving p and q derived from (7). The relation and equation (1) gives the value p and q . The values of p and q are substituted in (2). On integration new eq. (2) gives the solution of (1).

Example 11 : Solve $px + qy = pq$

Solution : $f(x, y, z, p, q) = 0$ is $px + qy - pq = 0$

$$\frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = x - q, \frac{\partial f}{\partial q} = y - p \quad \dots(1)$$

Charpits equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p} = \frac{dq}{q} = \frac{d\phi}{0}$$

We have choose the simplest integral involving p and q

or $\frac{dp}{p} = \frac{dq}{q}$ or $\log p = \log q + \log a \Rightarrow p = aq$

Putting for p in the given equation (1), we get

$$q(ax + y) = aq^2 \quad \therefore q = \frac{y+ax}{a}$$

$$\therefore p = aq = y + ax$$

$$\text{Now } dz = p dx + q dy$$

Putting for p and q in (2), we get

$$dz = (y + ax) dx + \frac{y + ax}{a} dy$$

$$a dz = (y + ax) dx + (y + ax) dy$$

$$a dz = (y + ax)(dx + dy)$$

$$a z = \frac{(y + ax)^2}{2} + b$$

Ans.

Integrating

Example 12 : Solve $(p^2 + q^2) = qz$

Solution : $f(x, y, z, p, q) = 0$ is $(p^2 + q^2)y - qz = 0$

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = p^2 + q^2, \frac{\partial f}{\partial z} = q, \frac{\partial f}{\partial p} = 2py, \frac{\partial f}{\partial q} = 2qy - z$$

Now Charpits equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

or $\frac{dy}{-2py} = \frac{dy}{-2q+q} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{d\phi}{0}$

We have to choose the simplest integral involving p and q

$$\frac{dp}{-pq} = \frac{dq}{p^2} \text{ or } \frac{dp}{p} = \frac{dq}{p} \text{ or } p dp + q dp = 0$$

Integrating

$$p^2 + q^2 = a^2 \text{ (say)}$$

Putting for $p^2 + q^2$ in the equation (1), we get

$$a^2 y = qz \text{ or } \frac{a^2 y}{z} \text{ so } p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \frac{a^4 y^2}{z^2}}$$

$$p = \frac{a}{2} \sqrt{z^2 - a^2 y^2}$$

Now

$$dz = p dx + q dy \quad \dots(2)$$

Putting for p and q (2), we get

$$dz = \frac{a}{z} \sqrt{z^2 - a^2 y^2} dx + \frac{a^2 y}{z} dy \text{ or } z dz = a \sqrt{z^2 - a^2 y^2} dx + a^2 y dy$$

$$\frac{z dz - a^2 y dy}{\sqrt{z^2 - a^2 y^2}} = a dx$$

$$\text{Integrating, we get } \frac{1}{2} \frac{2}{1} \sqrt{z^2 - a^2 y^2} = ax + b$$

$$(Put \ z^2 - a^2 y^2 = t)$$

On squaring,

$$z^2 - a^2 y^2 = (ax + b)^2$$

Ans.

Example 13 : Solve $(p+q)(px+qy)-1=0$

Solution : We have

$$f = (p+q)(px+qy)-1=0$$

Charpit's auxiliary equations are

$$\frac{dp}{p(p+q)+0} = \frac{dq}{q(p+q)+0} = \dots$$

From which it follows that

$$\frac{dp}{p} = \frac{dq}{q} \text{ or } p = aq \quad \dots(2)$$

Putting this value of p in (1), we get

$$q^2 (a+1)(ax+y) = 1$$

$$\therefore q = \frac{1}{\{(a+1)(ax+y)\}^{1/2}} \text{ and } p = \frac{a}{\{(a+1)(ax+y)\}^{1/2}}$$

Putting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \frac{a dx + dy}{\{(a+1)(ax+y)\}^{1/2}}$$

Integrating,

$$z = \sqrt{(a+1)} = 2\sqrt{(ax+y)} + b$$

Example 14 : Solve, $z = pq$

Solution : The given differential equation is

$$f = z - pq = 0 \quad \dots(1)$$

Charpit's auxiliary equation are

$$\frac{dp}{p} = \frac{dq}{q} = \dots$$

From which it follows that $p = aq$

...

Solving (1) and (2) simultaneously for p and q, we get

$$q = \sqrt{\left(\frac{z}{a}\right)} \text{ and } p = a \sqrt{\left(\frac{z}{a}\right)}$$

Putting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \sqrt{z} \left\{ \sqrt{a} dx + \frac{1}{\sqrt{a}} dy \right\}$$

$$\text{or } \frac{dz}{\sqrt{z}} = \sqrt{a} dx + \frac{1}{\sqrt{a}} dy$$

$$\text{Integrating } 2\sqrt{z} = \sqrt{a} \cdot x + \left(\frac{1}{\sqrt{a}}\right)y + b$$

Which is a complete integral of (1).

Example 15 : Solve $q + xp = p^2$

Solution : We have

$$q + xp = p^2$$

$$\text{or } f \equiv q + px - p^2 = 0$$

Charpit's auxiliary equation are

$$\frac{dp}{p} = \frac{dq}{0} = \dots\dots$$

From which it follows that $dq = 0$ or $q = a$

Putting $q = a$ in equation (1) we get

$$a + px - p^2 = 0 \quad \text{or} \quad p^2 - px - a = 0$$

$$\therefore p = \frac{1}{2} \left\{ x \pm \sqrt{x^2 + 4a} \right\}$$

Putting these values of p and q in $dz = p dx + q dy$, we get

$$\therefore dz = \frac{1}{2} \left\{ x \pm \sqrt{x^2 + 4a} \right\} dx + a dy$$

Integrating,

$$z = \frac{x^2}{4} \pm \frac{1}{2} \left[\frac{x}{2} \sqrt{x^2 + 4a} + 2a \log \left\{ x + \sqrt{x^2 + 4a} \right\} \right] + ay + b$$

Which is complete integral of (1).

Example 16 : Solve $z^2 = pq xy$

Solution : The given differential equation is

$$f \equiv z^2 - pq xy = 0 \quad \dots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dx}{qxy} = \frac{dy}{pxy} = \dots\dots$$

From which it follows that

$$\frac{xdp + p dx}{2pxz} = \frac{y dq + q dy}{2qyz}$$

$$\text{or } \frac{xdp + p dx}{px} = \frac{y dq + q dy}{qy}$$

Integrating, $\log px + \log qy + \log a^2$

$$\therefore px = a^2 qy \quad \dots(2)$$

Solving (1) and (2) simultaneously for p and q, we get

$$q = \frac{z}{ay} \quad \text{and} \quad p = \frac{az}{x}$$

Putting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \frac{az}{x} dx + \frac{z}{ay} dy \quad \text{or} \quad \frac{dz}{z} = \frac{a}{x} dx + \frac{1}{a} \frac{dy}{y}$$

Integrating, $\log z = a \log x + \frac{1}{a} \log y + \log b$

$$\text{or } z = bx^a y^{\frac{1}{a}}$$

Which is a complete integral of (1).

EXERCISE 2.4 :

Solve the following

- | | |
|-------------------------------|--------------------------------|
| (1) $z = p, q$ | (2) $(p + q)(px + qy) - 1 = 0$ |
| (3) $z = px + qy + p^2 + q^2$ | (4) $z = p^2x + q^2y$ |
| (5) $z^2 = pq, xy$ | (6) $px + pq + qy = yz$ |
| (7) $q + xp = p^2$ | (8) $q = 3p^2$ |

ANSWER 2.4 :

- | | |
|---|---|
| (1) $2\sqrt{az} = ax + y + \sqrt{ab}$ | (2) $z\sqrt{1+a} = 2\sqrt{(ax+y)+b}$ |
| (3) $z = ax + by + a^2 + b^2$ | (4) $(1+a)z = [\sqrt{ax} + \sqrt{(b+y)}]^2$ |
| (5) $z = ax^b y^{1/b}$ | (6) $\log(z - ax) = y - a \log(a + y) + b$ |
| (7) $z = ax e^{-y} - \frac{1}{2} a^2 e^{-2y} + b$ | (8) $z = ax + 3a^2 y + b$ |

2.3 LINEAR HOMOGENOUS PARTIAL DIFFERENTIAL EQUATION OF nth ORDER WITH CONSTANT COEFFICIENTS :

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad \dots(1)$$

is called a homogenous linear partial differential equation of nth order with constant coefficients. It is called homogenous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$, (1) becomes

$$(a_0 D^n + a_1 D^{n-1} D' + \dots + a_n D'^n) z = F(x, y)$$

$$f(D, D') z = F(x, y)$$



RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{or} \quad (a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0$$

 1st step : Put $D = m$ and $D' = 1$, $a_0 m^2 + a_1 m + a_2 = 0$

This is the auxiliary equation.

2nd step : Solve the auxiliary equation.

 Case 1 : If the roots of the auxiliary equation are real and different ; say m_1, m_2

$$\text{Then C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x)$$

 Case 2 : If the roots are equal say m

$$\text{Then C.F.} = f_1(y + m_1 x) + x f_2(y + m_2 x)$$

 Example 17 : Solve $(D^3 - 4D^2 D' + 3DD'^2) z = 0$

$$\text{Solution : } (D^3 - 4D^2 D' + 3DD'^2) z = 0 \quad [D = m, D' = 1]$$

 Its auxiliary equation is $m^3 - 4m^2 + 3m = 0$ or $m(m^2 - 4m + 3) = 0$

$$m(m-1)(m-3) = 0, m = 0, 1, 3$$

$$\text{The required solution is } z = f_1(y) + f_2(y+x) + f_3(y+3x)$$

Ans.

 Example 18 : Solve $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = 0$

Solution : $(D^2 - 4DD' + 4D'^2) = 0$

Its auxilliary equation is

$$[D = m, D' = 1]$$

$$m^2 - 4m + 4 = 0 \text{ or } (m-2)^2 = 0 \text{ or } m = 2, 2$$

The required solution is $z = f_1(y+2x) + xf_2(y+x)$ Ans.

Example 19 : Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

Solution : We have,

$$\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0 \dots\dots\dots$$

The equation (1) can be written as

$$(D^4 - D'^4)z = 0$$

Its auxilliary equation is

$$m^4 - 1 = 0$$

or $(m-1)(m+1)(m^2+1) = 0$

$\therefore m = 1, -1, \pm i$

$\therefore$ The general solution (1) is,

$$z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$$

Example 20: Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

Solution : We have

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$$

The auxilliary equation is

...(1)

$$\therefore m^3 - 6m^2 + 11m - 6 = 0$$

or $(m-1)(m-2)(m-3) = 0$

$$\therefore m = 1, 2, 3$$

The general solution of (1) is

$$\therefore z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$$

EXERCISE 2.5 :

Solve the following equations

(1) $\frac{\partial^2 z}{\partial x^2} + \frac{4\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$

(2) $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

(3) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

(4) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(5) $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

(6) $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

ANSWER 2.5 :

(1) $z = f_1(y+x) + f_2(y-5x)$

(2) $z = f_1(2y-x) + f_2(y-2x)$

(3) $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

(4) $z = f_1(y+x) + xf_2(y+x)$

(5) $z = f_1(y+2x) + xf_2(y+2x) + x^2f_3(y+2x)$

(6) $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$

2.4 * RULES FOR FINDING THE PARTICULAR INTEGRAL :

Given partial differential equation is

$$f(D, D')Z = F(x, y)$$

$$P.I. = \frac{1}{f(D, D')} F(x, y)$$

(i) when $F(x, y) = e^{ax+by}$ $P.I. = \frac{1}{f(D, D')} e^{ax+by}$

$$= \frac{e^{ax+by}}{f(a, b)}$$

[Put $D = a, D' = b$]

(ii) When $F(x, y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Put

$$D^2 = -a^2$$

$$DD' = -ab$$

$$D'^2 = -b^2$$

$$= \frac{\sin(ax+by) \text{ or } \cos(ax+by)}{f(-a^2, -ab, -b^2)}$$

(iii) When $F(x, y) = x^m y^n$

$$P.I. = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expan $[f(D, D')]^{-1}$ is ascending power of D or D' and operate on $x^m y^n$ term by term

(iv) When = Any function $F(x, y)$

$$P.I. = \frac{1}{f(D, D')} F(x, y)$$

Resolve $\frac{1}{f(D, D')}$ into partial fractions. Considering $f(D, D')$ as a function

of D alone $P.I. = \frac{1}{D - mD'} F(x, y) = \int f(x, c - mx) dx$ when c replaced by $y + mx$

after into

Case 1 : When R.H.S. = e^{ax+by}

Example 21 : Solve $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^2 z}{\partial y^3} = e^{x+2y}$

Solution : $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^2 z}{\partial y^3} = e^{x+2y}$

Given equation in symbolic form is

$$(D^3 - 3D^2 D' + 4D'^3)z = e^{x+2y}$$

Its A.E. is $m^3 - 3m^2 + 4 = 0$ whence, $m = -1, 2, 2$

$$C.F. = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I. = \frac{1}{D^3 - 3D^2 D' + 4D'^3} e^{x+2y}$$

Put $D = 1, D' = 2$ $= \frac{1}{1 - 6 + 32} e^{x+2y} = \frac{e^{x+2y}}{27}$

Hence complete solution is

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Ans.

EXERCISE 2.6 :

(1) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$

(3) $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

(5) $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x-y}$

(7) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial z}{\partial y^2} = e^{2x+3y}$

(2) $\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = e^{x+y}$

(4) $\frac{\partial^2 z}{\partial x^2} - 7 \frac{\partial^2 z}{\partial x \partial y} + 12 \frac{\partial^2 z}{\partial y^2} = e^{x-y}$

(6) $(D^2 - 2DD' + D'^2)z = e^{x+2y}$

(8) $\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = \exp(3x-2y)$

ANSWER 2.6 :

(1) $z = f_1(y+x) + f_2(y-x) - \frac{e^{x+2y}}{3}$

(2) $z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2}e^{x+y}$

(3) $z = f_1(y+2x) + xf_2(y+2x) + \frac{x^2}{2}e^{2x+y}$

(4) $z = f_1(y+3x) + f_2(y+4x) + \frac{1}{20}e^{x-y}$

(5) $z = f_1(y) + xf_2(y) + f_3(y+2x) + \frac{1}{8}e^{2x-y}$

(6) $z = f_1(y+x) + xf_2(y+x) + e^{x-2y}$

(7) $z = f_1(y+x) + e^{2x}f_2(y-x) - \frac{1}{3}e^{2x+3y}$

(8) $z = f_1(y+2x) + f_2(y+3x) + \frac{1}{63}e^{3x+2y}$

Case II : When R.H.S. = $\sin(ax+by)$ or $\cos(ax+by)$

Example 22 : Solve $\frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 2 \sin(3x+2y)$

Solution : $\frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 2 \sin(3x+2y)$

Putting $\frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$

$$D^3 z = 4D^2 D' z + 4DD'^2 z = 2 \sin(3x+2y)$$

A.E. is $D^3 - 4D^2 D' + 4DD'^2 = 0$ or $D(D^2 - 4DD' + 4D'^2) = 0$

Put $D = m, D' = 1$

$$m(m^3 - 4m + 4) = 0 \text{ or } m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$$

C.F. is $f_1(y) + f_2(y+2x) + xf_3(y+2x)$

$$P.I. = \frac{1}{D^3 - 4D^2 D' + 4DD'^2} 2 \sin(3x+2y)$$

$$= 2 \frac{1}{D(D^2 - 4DD' + 4D'^2)} \sin(3x+2y)$$

$$= 2 \frac{1}{D[-9 - 4(-6) + 4(-4)]} \sin(3x+2y) = -\frac{2}{D} \sin(3x+2y)$$

$$= -\frac{2}{3} [-\cos(3x+2y)] = \frac{2}{3} \cos(3x+2y)$$

General solution is

$$z = f_1(y) + f_2(y+2x) + xf_3(y+2x) + \frac{2}{3}\cos(3x+2y)$$

Example 23 : Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - \sin x \cos 2y$

Solution : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - \sin x \cos 2y$

The given equation can be written in the form

$$(D^2 - DD')z = \sin x \cos 2y$$

where $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

Writing $D = m$ and $D' = 1$, the auxillary equation is

$$m^2 - m = 0 \quad \text{or} \quad m(m-1) = 0, \quad m = 0, 1$$

$$C.F. = f_1(y) + f_2(y+x)$$

$$P.I. = \frac{1}{D^2 - DD'} \sin x \cos 2y$$

$$= \frac{1}{D^2 - DD'} \cdot \frac{1}{2} [\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x+2y) + \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x-2y)$$

Put $D^2 = -1, DD' = 2$ in the first integral and $D^2 = -1, DD' = 2$ in the second integral.

$$= \frac{1}{2} \frac{\sin(x+2y)}{-1-(-2)} + \frac{1}{2} \frac{\sin(x-2y)}{-1-(-2)} = \frac{1}{2} \sin(x+2y) - \frac{1}{6} \sin(x-2y)$$

Hence the complete solution is $z = C.F. + P.I.$

ie. $z = f_1(y) + f_2(y+x) + \frac{1}{2} \sin(x+2y) - \frac{1}{6} \sin(x-2y)$

Ans.

Example 24 : Solve $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

Solution : $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

A.E. is $m^2 + m - 6 = 0$ or $m = 2, -3$

$$C.F. = f_1(y+2x) + f_2(y-3x)$$

$$P.I. = \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$$

$$D^2 + DD' - 6D'^2 = -4 - 2 - 6(-1) = 0$$

It is case of failure.

Now $P.I. = \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$

Case IV : $x \frac{1}{2D + D'} \cos(2x+y) = x \frac{D}{2D^2 + DD'} \cos(2x+y)$

$$x = \frac{D}{2(-4) - 2} \cos 2(2x+y) = -\frac{x}{10} D \cos(2x+y)$$

$$= 2 \frac{x}{10} \sin(2x+y) = \frac{x}{5} \sin(2x+y)$$

$$z = f_1(y+2x) + f_2(y-3x) + \frac{x}{5} \sin(2x+y)$$

Ans.

Example 25 : Solve the equation $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

Solution : $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

Its auxillary equation is

$$m^3 - 7m - 6 = 0 \quad \text{or} \quad (m+1)(m+2)(m-3) = 0 \Rightarrow m = -1, -2, 3$$

$$C.F. = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$$

$$P.I. = \frac{1}{D^3 - 7DD'^2 - 6D'^3} [\sin(x+2y) + e^{2x+y}]$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{2x+y}$$

$$= \frac{1}{D^2 - 7DD' - 6D'^2} \sin(x+2y) + \frac{e^{2x+y}}{(2)^3 - 7(2)(1)^2 - 6(1)^3}$$

Put $D^2 = -1, D'^2 = -2^2$

$$= \frac{1}{-D - 7D(-4) - 6(-4)D'} \sin(x+2y) + \frac{e^{2x+y}}{8-14-6}$$

$$= \frac{1}{27D + 24D'} \sin(x+2y) - \frac{1}{12} e^{2x+y} = \frac{1}{3 \cdot 9D + 8D'} \sin(x+2y) - \frac{1}{12} e^{2x+y}$$

$$= \frac{1}{3 \cdot 9D^2 + 8DD'} \sin(x+2y) - \frac{1}{12} e^{2x+y} = \frac{1}{3 \cdot 9(-1) + 8(-2)} \sin(2x+y) - \frac{1}{12} e^{2x+y}$$

$$= -\frac{1}{75} D \sin(x+2y) - \frac{1}{12} e^{2x+y} = -\frac{1}{75} \cos(x+2y) - \frac{1}{12} e^{2x+y}$$

hence The complete solution is

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75} \cos(x+2y) - \frac{1}{12} e^{2x+y}$$

EXERCISE 2.7 :

Solve the following.

$$1. \quad \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x \quad 2. \quad [2D^2 - 5DD' + 2D'^2]z = 5 \sin(2x+y)$$

$$3. \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos(x+2y) \quad 4. \quad (D^2 - DD')z = \cos x \cos 2y$$

$$5. \quad (D^2 + 2D' + D'^2)z = \sin(x+2y)$$

$$6. \quad \frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x+2y)$$

ANSWER 2.7 :

$$(1) \quad z = f_1(y+x) + x f_2(y+x) - \sin x$$

$$(2) \quad z = f_1(y+2x) + f_2(2y+x) - \frac{5}{3} x \cos(2x+y)$$

$$(3) \quad z = f_1(y) + f_2(y+x) + \cos(x+2y)$$

$$(4) \quad z = f_1(y) + f_2(y+x) + \frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x-2y)$$

$$(5) \quad z = f_1(y-x) + x f_2(y-x) - \frac{1}{9} \sin(x-2y)$$

$$(6) \quad z = f_1(y+x) + f_2(y+2x) + \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x-2y)$$

Case III : When R.H.S. = $x^m y^n$

Example 26 : Find the general integral of the equation

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$$

Solution : $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

with $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$, the given equation can be written in the form

$$(D^2 + 3DD' + 2D'^2)z = x + y$$

Writing $D = m$ and $D' = 1$, the auxillary equation is

$$m^3 + 3m + 2 = 0 \text{ or } (m+1)(m+2) = 0, m = -1, -2$$

$$C.F. = f_1(y-x) + f_2(y-2x)$$

$$P.I. = \frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$$

$$= \frac{1}{D^2} \left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2} \right) (x+y) = \frac{1}{D^2} \left(1 - \frac{3D'}{D} + \dots \right) (x+y)$$

$$= \frac{1}{D^2} \left[x+y - 3\frac{1}{D}(1) \right] = \frac{1}{D^2} [x+y-3x]$$

$$= \frac{1}{D^2} [y-2x] = \frac{x^2}{2}y - \frac{x^3}{3}$$

Hence the complete solution is $z = f_1(y-x) + f_2(y-2x) + \frac{x^2y}{2} - \frac{x^3}{3}$

Ans.

Example 27 : Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x + y$

Solution : with $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$ the given equation can be written in the form

$$(D^2 + DD' - 6D'^2)z = x + y$$

Writing $D = m$ and $D' = 1$, the auxillary equation is $m^2 + m - 6 = 0$

or $(m+3)(m-2) = 0 \Rightarrow m = -3, 2$

$\therefore C.F. = f_1(y-3x) + f_2(y+2x)$

$\therefore P.I. = \frac{1}{D^2 + DD' - 6D'^2}(x+y)$

$$= \frac{1}{D^2} \left(1 + \frac{D'}{D} - \frac{6D'^2}{D^2} \right)^{-1} (x+y) = \frac{1}{D^2} \left[1 - \frac{D'}{D} + \dots \right] (x+y)$$

$$= \frac{1}{D^2} \left[x+y - \frac{1}{D}(1) \right] = \frac{1}{D^2} (x+y-x) = \frac{1}{D^2} y = \frac{yx^2}{2}$$

The complete solution is

$$z = f_1(y-3x) + f_2(y+2x) + \frac{yx^2}{2}$$

Ans.

Example 28 : Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

Solution : $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

It auxillary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m - 2) = 0$$

$$m = 0, 0, 2$$

$$C.F = f_1(y) + xf_2(y) + f_3(y + 2x)$$

$$P.I. = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y)$$

$$= \frac{1}{D^3 - 2D^2D'}2e^{2x} + \frac{1}{D^3 - 2D^2D'}3x^2y$$

$$= 2 \frac{e^{2x}}{(2)^3 - 2(2)^2(0)} + 3 \frac{1}{D^3 \left(1 - \frac{2D'}{D}\right)} x^2y = \frac{2e^{2x}}{8} + \frac{3}{D^3} \left(1 - \frac{2D'}{D}\right) x^2y$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3} \left(1 + \frac{2D'}{D} \dots\right) x^2y = \frac{e^{2x}}{4} + \frac{3}{D^3} \left[x^2y + \frac{2}{D}x^2\right] = \frac{e^{2x}}{4} + \frac{3}{D^3} \left(x^2y + \frac{2x^3}{3}\right)$$

$$= \frac{e^{2x}}{4} + 3y \frac{1}{D^3}x^2 + \frac{2}{D^3}x^3 = \frac{e^{2x}}{4} + 3y \frac{x^5}{3.4.5} + 2 \frac{x^6}{4.5.6} = \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$

$$= \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$$

Hence the complete solution is

$$z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$$

Ans.

EXERCISE 2.8 :

Solve the following.

$$1. \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

$$2. \quad \frac{\partial^2 z}{\partial x^2} + \frac{3\partial^2 z}{\partial x \partial y} + \frac{2\partial^2 z}{\partial y^2} = 12xy$$

$$3. \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = xy$$

$$4. \quad r + 2s + t = 2(y - x) + \sin(x - y)$$

$$5. \quad \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = x^2$$

$$6. \quad \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$$

$$7. \quad \frac{\partial^2 z}{\partial x^2} + 3 \frac{\partial^2 z}{\partial x \partial y} - 4 \frac{\partial^2 z}{\partial y^2} = x + \sin y$$

$$8. \quad (D^3 - 3D^2D')z = x^2y$$

ANSWER 2.8 :

$$(1) \quad \text{Ans. } z = f_1(y - x) + f_2(y + x) + \frac{x^3}{6} - \frac{x^2y}{2}$$

$$(2) \quad \text{Ans. } z = f_1(y - x) + f_2(y - 2x) + 2x^3y - \frac{3x^4}{2}$$

$$(3) \quad \text{Ans. } z = f_1(y - 2x) + f_2(y + 3x) + \frac{x^3y}{6} + \frac{x^4}{24}$$

$$(4) \quad \text{Ans. } z = f_1(y - x) + xf_2(y - x) + x^2y - x^3 + \frac{x^2}{2} \sin(x - y)$$

$$(5) \quad \text{Ans. } z = f_1(y + ax) + f_2(y - ax) + \frac{x^4}{12}$$

$$(6) \quad \text{Ans. } z = f_1(y + x) + xf_2(y + x) + \frac{x^4}{12} + \frac{x^2y}{2} + \frac{x^3}{13}$$

$$(7) \quad \text{Ans. } z = f_1(y+x) + f_2(y-4x) + \frac{x^3}{6} + \frac{1}{4} \sin y$$

$$(8) \quad \text{Ans. } z = f_1(y) + x f_2(y) + f_3(y+3x) + \frac{x^5 y}{60} + \frac{x^4}{120}$$

Case IV : When R.H.S. = Any function

Example 29 : Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution : $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

A.E. is

$$D^2 - DD' - 2D'^2 = 0 \quad \text{or} \quad m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0, \quad \text{or} \quad m = 2, -1$$

$$C.F. = f_1(y+2x) + f_2(y-x)$$

$$P.I. = \frac{1}{D^2 - DD' - 2D'^2} (y-1)e^x$$

$$= \frac{1}{(D+D')(D-2D')} (y-1)e^x = \frac{1}{D+D'} \int [(c-2x-1)e^x dx]$$

Put $y = c - 2x$

$$= \frac{1}{D+D'} [(c-2x-1)e^x + 2e^x]$$

$$= \frac{1}{D+D'} [ce^x - 2xe^x + e^x]$$

[Put $c = y + 2x$]

$$= \frac{1}{D+D'} [(y+2x)e^x - 2xe^x + e^x] = \frac{1}{D+D'} [ye^x + e^x]$$

$$= \int [(c+x)e^x + e^x] dx$$

Put $y = c + x$

$$= (c+x)e^x - e^x + e^x$$

$$= ce^x + xe^x = (y-x)e^x + xe^x$$

Put $c = y - x$

$$= ye^x$$

Hence complete solution is

$$z = f_1(y+2x) + f_2(y-x) + ye^x$$

Ans.

Example 30 : Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$

Solution : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

It auxillary equation is,

$$m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$m = 2, -3$$

$$C.F. = f_1(y+2x) + f_2(y-3x)$$

$$P.I. = \frac{1}{D^2 + DD' - 6D'^2} y \cos x = \frac{1}{(D-2D')(D+3D')} y \cos x$$

$$= \frac{1}{D-2D'} \int (c+3x) \cos x dx$$

Put $y = c + 3x$

$$= \frac{1}{D-2D'} [(c+3x) \sin x + 3 \cos x] = \frac{1}{D-2D'} [y \sin x + 3 \cos x]$$

Put

$$c + 3x = y$$

$$= \int [(c - 2x) \sin x + 3 \cos x] dx$$

$$= (c - 2x)(-\cos x) - 2 \sin x + 3 \sin x = -y \cos x + \sin x$$

$$c - 2x = y$$

Hence the complete solution is

$$z = f_1(y + 2x) + f_2(y - 3x) + \sin x - y \cos x$$

Put $y = c - 2x$

Ans.

EXERCISE 2.9 :

Solve the following.

$$1. (D - D')(D + 2D')z = (y + 1)e^x$$

$$2. \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \tan^3 x \tan y - \tan x \tan^3 y$$

$$3. (D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy$$

ANSWER 2.9 :

$$(1) z = f_1(y + x) + f_2(y - 2x) + ye^x$$

$$(2) z = f_1(y + x) + f_2(x - y) + \frac{1}{2} \tan x \tan y$$

$$(3) z = f_1(y + 2x) + f_2(y - x) + \sin xy$$

2.5 NON-HOMOGENEOUS LINEAR EQUATIONS :

The linear different equations which are not homogenous are called Non-homogeneous linear equations.

For example,

$$3 \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} + 5 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0$$

$$f(D, D') = f_1(x, y)$$

Its solution, $z = C.F. + P.I.$

Complementary Function: Let the non-homogenous equation be

$$(D - mD' - a)z = 0 \text{ or } \frac{\partial z}{\partial x} - m \frac{\partial z}{\partial y} - az = 0$$

$$p - mq = az$$

The Languages' subsidiary equations are, ..

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

From first two relations we have, $m dx = dy$

$$dy + m dx = 0 \Rightarrow y + mx = c_1 \quad \dots(1)$$

$$\text{and from first and third relation, } dx = \frac{dz}{az} \Rightarrow x = \frac{1}{a} \log z + c_2 \Rightarrow z = c_3 e^{ax} \quad \dots(2)$$

From (1) and (2), we have $z = e^{ax} \phi(y + mx)$

Similarly the solution of $(D - mD' - a)^2 z = 0$ is

$$z = e^{ax} \phi_1(y + mx) + x e^{ax} \phi_2(y + mx)$$

Example 31 : Solve $(D + D' - 2)(D + 4D' - 3)z = 0$

Solution : The equation can be rewritten as,

$$\{D - (-D)' - 2\} \{D(-4D)' - 3\} z = 0$$

Hence the solution is

$$z = e^{2x} \phi_1(y - mx) + e^{3x} \phi_2(y - 4mx) \quad \text{Ans.}$$

Example 32 : Solve $(D + 3D' + 4)^2 z = 0$

Solution : The equation is rewritten as

$$[D - (-3D') - (-4)^2]z = 0$$

Hence the solution is

$$z = e^{-4x} \phi_1(y - 3x) + x e^{-4x} \phi_2(y - 3x)$$

Ans.

Example 33 : Solve $r + 2s + t + 2p + 2q + z = 0$

Solution : The equation is rewritten as

$$(D^2 + 2D D' + D^2 + 2D + 2D' + 1)z = 0$$

$$\Rightarrow [(D + D')^2 + 2(D + D') + 1]z = 0$$

$$\Rightarrow (D + D' + 1)^2 z = 0$$

$$\Rightarrow [D - (-D') - (-1)]^2 z = 0$$

Hence the solution is

$$z = e^{-x} \phi_1(y - x) + x e^{-x} \phi_2(y - x)$$

Example 34 : Solve $r - t + p - q = 0$

Solution : The equation is rewritten as

$$(D^2 - D'^2 + D - D')z = 0$$

$$\Rightarrow [(D - D')(D + D') + 1(D - D')]z = 0$$

$$\Rightarrow (D - D')(D + D' + 1)z = 0$$

Hence the solution is

$$z = \phi_1(y + x) + e^{-x} \phi_2(y - x)$$

Ans.

2.6 PARTICULAR INTEGRAL

Case 1 : $\frac{1}{F(D - D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$

Example 35: Solve $(D - D' - 2)(D - D' - 3)z = e^{3x-2y}$

The complementary solution is

$$e^{2x} \phi_1(y + x) + e^{3x} \phi_2(y + x)$$

$$P.I. = \frac{1}{(D - D' - 2)(D - D' - 3)} e^{3x-2y} = \frac{1}{[3 - (-2) - 2][3 - (-2) - 3]} e^{3x-2y} = \frac{1}{6} e^{3x-2y}$$

Hence the complete solution is

$$z = e^{2x} \phi_1(y + x) + e^{3x} \phi_2(y + x) + \frac{1}{6} e^{3x-2y}$$

Ans.

Case 2 : $\frac{1}{F(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by)$

Example 36 : Solve $(D + 1)(D + D' - 1)z = \sin(x + 2y)$

Solution : C.F. = $e^{-x} \phi(y) + e^x \phi_2(y - x)$

$$P.I. = \frac{1}{(D + 1)(D + D' - 1)} \sin(x + 2y) = \frac{1}{D^2 + DD' + D' - 1} \sin(x + 2y)$$

$$= \frac{1}{-1 + (-2) + D' - 1} \sin(x + 2y) = \frac{1}{D' - 4} \sin(x + 2y)$$

$$= \frac{D' + 4}{(D'^2 - 16)} \sin(x + 2y) = \frac{D' + 4}{(-4 - 16)} \sin(x + 2y)$$

$$= -\frac{1}{20}(D'+4)\sin(x+2y) = -\frac{1}{20}[D'\sin(x+2y) + 4\sin(x+2y)]$$

$$= -\frac{1}{20}[2\cos(x+2y) + 4\sin(x+2y)]$$

Hence, the solution is...

Ans.

$$z = e^{-x}\phi(y) + e^x\phi_2(y-x) - \frac{1}{10}[\cos(x+2y) + 2\sin(x+2y)]$$

Case 3 : $\frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$

Example 37 : Solve $[D^2 - D'^2 + D + 3D' - 2]z = x^2 y$

Solution : $(D - D' + 2)(D + D' - 1)z = 0$

$$C.F. = e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$$

$$P.I. = \frac{1}{(D - D' + 2)(D + D' - 1)} x^2 y$$

$$= \frac{1}{D^2 - D'^2 + 3D' - 2} x^2 y = -\frac{1}{2} \frac{1}{1 - \frac{3D'}{2} - \frac{D}{2} + \frac{D'^2}{2} - \frac{D^2}{2}} x^2 y$$

$$= -\frac{1}{2} \left[1 - \frac{1}{2}(3D' + D - D'^2) + D^2 \right]^{-1} x^2 y$$

$$= -\frac{1}{2} \left[1 + \frac{1}{2}(3D' + D - D'^2 + D^2) + \frac{1}{4}(3D' + D - D'^2 + D^2)^2 + \frac{1}{8}(3D' + D - D'^2 + D^2)^3 \right] x^2 y$$

$$= -\frac{1}{2} \left[1 + \frac{1}{2}(3D' + D - D'^2 + D^2) + \frac{1}{4}(9D'^2 + D^2 + 6DD' + 6D^2 D') + \frac{1}{8}(9D^2 D') \dots \right] x^2 y$$

$$= -\frac{1}{2} \left[x^2 y + \frac{1}{2}(3x^2 + 2xy - 0 + 2y) + \frac{1}{4}(0 + 2y + 12x + 12) + \frac{1}{8}(18) \right]$$

$$= -\frac{1}{2} \left[x^2 y + \frac{3x^2}{2} + xy + y + \frac{y}{2} + 3x + 3 + \frac{9}{4} \right] = -\frac{1}{2} \left[x^2 y + \frac{3x^2}{2} + xy + \frac{3y}{2} + 3x + \frac{21}{4} \right]$$

Solution is

$$z = e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x) - \frac{1}{2} \left(x^2 y + \frac{3x^2}{2} + xy + \frac{3y}{2} + 3x + \frac{21}{4} \right) \quad \text{Ans.}$$

Case 4 : $\frac{1}{F(D, D')} [e^{ax+by} \phi(x, y)]$ of x and y

$$\frac{1}{F(D, D')} [e^{ax+by} \phi(x, y)] = e^{ax+by} \frac{1}{F(D+a, D'+b)} \phi(x, y)$$

Example 38 : Solve $(D - 3D' - 2)^2 z = 2e^{2x} \sin(y + 3x)$

Solution : A.E. is $(D - 3D' - 2)^2 = 0$

$$C.F. = e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x)$$

$$P.I. = \frac{1}{(D - 3D' - 2)^2} 2e^{2x} \sin(y + 3x)$$

$$= 2e^{2x} \frac{1}{(D + 2 - 3D' - 2)^2} \sin(y + 3x) = 2e^{2x} \frac{1}{(D - 3D')^2} \sin(y + 3x)$$

$$= 2e^{2x} \cdot x \frac{1}{2(D - 3D')} \sin(y + 3x) \quad (\text{As denominator becomes zero})$$

$$= 2x^2 e^{2x} \frac{1}{2} \sin(y+3x) \quad (\text{Again differentiate})$$

$$= x^2 e^{2x} \sin(y+3x)$$

Hence the complete solution is

$$z = e^{2x} \phi_1(y+3x) + x e^{2x} \phi_2(y+3x) + x^2 e^{2x} \sin(y+3x)$$

Ans.

Example 39 : Solve $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$

Solution : $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$

For complementary function

$$(D^2 + DD' - 6D'^2) = 0 \quad \text{or} \quad (D - 2D')(D + 3D') = 0$$

$$C.F. = \phi_1(y+2x) + \phi_2(y-3x)$$

$$P.I. = \frac{1}{D^2 + DD' - 6D'^2} x^2 \sin(x+y)$$

$$= \text{Imaginary part of } \frac{1}{D^2 + DD' - 6D'^2} x^2 [\cos(x+y) + i \sin(x+y)]$$

$$= " \frac{1}{D^2 + DD' - 6D'^2} x^2 e^{i(x+y)}$$

$$= " e^{iy} \frac{1}{D^2 + Di - 6(i)^2} x^2 e^{ix}$$

$$= " e^{i(x+y)} \frac{1}{(D+i)^2 + (D+i)i + 6} x^2$$

$$= " e^{i(x+y)} \frac{1}{D^2 + 3iD + 4} x^2$$

$$= " \frac{e^{i(x+y)}}{4} \frac{1}{1 + \frac{3iD}{4} + \frac{D^2}{4}} x^2$$

$$= " \frac{e^{i(x+y)}}{4} \left[1 + \frac{3iD}{4} + \frac{D^2}{4} \right]^{-1} x^2$$

$$= " \frac{e^{i(x+y)}}{4} \left[1 - \frac{3iD}{4} - \frac{D^2}{4} - \frac{9D^2}{16} \dots \right] x^2$$

$$= " \frac{e^{i(x+y)}}{4} \left[x^2 - \frac{3ix}{2} - \frac{2}{4} - \frac{9}{16}(2) \right]$$

$$= " \frac{1}{4} [\cos(x+y) + i \sin(x+y)] \left[x^2 - \frac{3ix}{2} - \frac{13}{8} \right]$$

$$= \frac{1}{4} \left[\sin(x+y) \left(x^2 - \frac{13}{8} \right) - \frac{3}{2} x \cos(x+y) \right] = \frac{1}{4} \sin(x+y) \left(x^2 - \frac{13}{8} \right) - \frac{3x}{8} \cos(x+y)$$

Hence, the complete solution is

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \frac{1}{4} \sin(x+y) \left(x^2 - \frac{13}{8} \right) - \frac{3x}{8} \cos(x+y)$$

Ans.

EXERCISE 2.10 :

Solve the following equations.

1. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = 0$
2. $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$
3. $(D - D' - 1)(D + D' - 2)z = e^{2x-y}$
4. $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$
5. $(D + D')(D + D' - 2)z = \sin(x+2y)$
6. $(D^2 - DD' - 2D)z = \cos(3x+4y)$

7. $(DD' + D - D' - 1)z = xy$ 8. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
9. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = xy + e^{x+2y}$ 10. $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$
11. $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$ 12. $(D - D' + 2)(D + D' - 1)z = e^{x-y} - x^2y$

ANSWER 2.10 :

- (1) $z = f_1(x - y) + e^{2x}f_2(x - y)$
- (2) $z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - \frac{1}{12}e^{x-2y}$
- (3) $z = e^x\phi_1(x + y) + e^{2x}\phi_1(y - x) - \frac{1}{12}e^{2x-y}$
- (4) $z = \phi_1(y + x) + e^{3x}\phi_1(x - y) - xe^{x+2y}$
- (5) $z = \phi_1(y - x) + e^{2x}(y - x) + \frac{1}{117}[6\cos(x + 2y) - 9\sin(x + 2y)]$
- (6) $z = \phi_1(y) + e^{2x}\phi_2(y + x) + \frac{1}{15}[\cos(3x + 4y) - 2\sin(3x + 4y)]$
- (7) $z = e^{-y}\phi_1(x) + e^x\phi_2(y) - (xy + y - x - 1)$
- (8) $z = e^x\phi_1(x - y) + e^{3x}\phi_2(2x - y) + 6 + x + 2y$
- (9) $z = f_1(y + x) + e^{3x}f_2(y - x) - \frac{1}{3}\left(\frac{x^2y}{2} + \frac{x^3}{6} + \frac{x^2}{3} + \frac{xy}{3} + \frac{2x}{9}\right) - xe^{x+2y}$
- (10) $z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{2}e^{2x-y}$
- (11) $z = \phi_1(y) + e^x\phi_2(x - y) + e^{2x}\phi_3(3x - y) + \frac{1}{2}\left[\frac{x^3}{3} - 2x^2y + 2xy^2 - \frac{7}{2}x^2 + 4xy + \frac{x}{2}\right]$

$$(12) \quad z = e^{2y}\phi_1(x + y) + e^x\phi_2(x - y) - \frac{e^{x-y}}{4} + \frac{1}{2}\left[x^2y + xy + \frac{3x^2}{2} + \frac{3}{2}y + 3x + \frac{21}{4}\right]$$

2.7 MONGES METHOD (Non linear equation of the second order) :

Let the equation be $Rr + Ss + Tt = v$... (1)

Where R, S, T, V are function of x, y, z, p and q , $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

we have $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy$... (2)

and $dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy$... (3)

From (2) and (3), we have $r = \frac{dp - sdy}{dx}$ and $t = \frac{dq - sdx}{dy}$

Putting the value of r and t in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

or $Rdp dy + Tdq dx - V dx dy - s(Rdy^2 - S dx dy + T dx^2) = 0$... (4)

Equation (4) is satisfied if

$$R dp dy + T dq dx - V dx dy = 0$$

$$R dy^2 - S dx dy + T dx^2 = 0$$

Equations (5) and (6) are called Monge's equations.

Since (6) can be factorised into two equations

$$dy - m_1 dx = 0 \text{ and } dy - m_2 dx = 0$$

Now combine $dy - m_1 dx = 0$ and equation (5). If need be, we may also use the relation

$dz = p dx + q dy$ while solving (5) and (6). The solution leads to two integrals

between U and V $u(x, y, z, p, q) = a$ and $V(x, y, z, p, q) = b$ then we get relation

$$V = f_1(u) \quad \dots(7)$$

Equation (7) is further integrated by methods of first order equations.

Note : If the intermediate solution is of the form $Pr + Qq = R$, then we use Lagrange's equations.

Example 40 : Solve $r = a^2 t$

Solution : We have $dp = r dx + s dy$ and $dq = s dx + t dy$ which gives

$$r = \frac{dp - s dy}{dx} \text{ and } t = \frac{dq - s dx}{dy}$$

Putting these values of r and t in $r = a^2 t$, we get

$$\frac{dp - s dy}{dx} = a^2 \frac{dq - s dx}{dy}$$

$$\text{or } dp dy - a^2 dx dq - s(dy^2 - a^2 dx^2) = 0$$

Thus, the Monge's equations are

$$dp dy - a^2 dx dq = 0$$

$$dy^2 - a^2 dx^2 = 0 \quad \dots(1)$$

(2) can be resolved into factors

$$dy - a dx = 0 \quad \dots(2)$$

and

$$dy + a dx = 0 \quad \dots(3)$$

combining (3) with (1) we get

$$dp(adx) - a^2 dx dq = 0 \text{ or } dp - adq = 0 \quad \dots(4)$$

$$\dots(5)$$

(3) and (5) on integration gives respectively

$$\left. \begin{array}{l} y - ax = A \\ \text{and } p - aq = B \end{array} \right\} \Rightarrow p - aq = f_1(y - ax) \quad \dots(6)$$

Similarly combining (4) and (1)

$$p + aq = f_2(y + ax) \quad \dots(7)$$

Adding and subtracting (6) and (7), we get

$$p = \frac{1}{2} [f_1(y - ax) + f_2(y + ax)], \quad q = \frac{1}{2a} [f_2(y + ax) - f_1(y - ax)]$$

Substituting these values in $dz = p dx + q dy$

$$dz = \frac{1}{2} [f_1(y - ax) + f_2(y + ax)] dx + \frac{1}{2a} [f_2(y + ax) - f_1(y - ax)] dy$$

$$dz = \frac{1}{2a} (dy + a dx) f_2(y + ax) - \frac{1}{2a} (dy - a dx) f_1(y - ax)$$

Integrating

$$z = \frac{1}{2a} \phi_1(y + ax) - \frac{1}{2a} \phi_2(y - ax)$$

or

$$z = F_1(y + ax) + F_2(y - ax) \quad \text{Ans.}$$

Example 41 : Solve $r - t \cos^2 x + p \tan x = 0$

Solution : $r = \frac{dp - s dy}{dx}$ and $t = \frac{dq - s dx}{dy}$

Putting for r and t in the given equation, we get

$$\frac{dp - s dy}{dx} - \frac{dq - s dx}{dy} \cos^2 x + p \tan x = 0$$

or

$$dp dy - s dy^2 - dx dq \cos^2 x + s dx^2 \cos^2 x + p dx dy \tan x = 0$$

or

$$dp dy - dx dq \cos^2 x + p dx dy \tan x - s(dy^2 - dx^2 \cos^2 x) = 0$$

Monge's equations are

$$dp dy - dx dq \cos^2 x + p dx dy \tan x = 0 \quad \dots(1)$$

$$dy^2 - dx^2 \cos^2 x = 0 \quad \dots(2)$$

Eq. (2) is factorised $(dy + dx \cos x)(dy - dx \cos x) = 0$

$$dy - dx \cos x = 0 \quad \dots(3)$$

$$dy + dx \cos x = 0 \quad \dots(4)$$

Integrating (3) and (4), we get

$$y - \sin x = A \quad \dots(5)$$

$$y + \sin x = B \quad \dots(6)$$

Combining (3) and (1), we get

$$dp - dq \cos x + p \tan x dx = 0$$

or

$$(dp \sec x + p \sec x \tan x dx) - dq = 0$$

Integrating

$$p \sec x - q = B$$

Combining (5) and (7), we have

$$p \sec x - q = f_1(y - \sin x) \quad \dots(8)$$

In combining (6) and (7), we get

$$p \sec x + q = f_2(y + \sin x) \quad \dots(9)$$

From (5) and (9)

$$p = \frac{1}{2} \cos x [f_1(y - \sin x) + f_2(y + \sin x)] \text{ and } q = \frac{1}{2} [f_2(y + \sin x) - f_1(y - \sin x)]$$

putting for p and q in

or

$$dz = p dx + q dy$$

$$dz = \frac{1}{2} \cos x [f_1(y - \sin x) + f_2(y + \sin x)] dx + \frac{1}{2} [f_2(y + \sin x) - f_1(y - \sin x)] dy$$

$$dz = \frac{1}{2} f_2(y + \sin x) [dy + \cos x dx] - \frac{1}{2} f_1(y - \sin x) [dy - \cos x dx]$$

$$\text{Integrating we get, } z = \frac{1}{2} F_2(y + \sin x) + F_1(y - \sin x) \quad \text{Ans.}$$

Example 42 : Solve $r + (a+b)s + abt = xy$

Solution : We have

$$r + (a+b)s + abt = xy \quad \dots(1)$$

$$\text{Now } dp = r dx + s dy \text{ and } dq = s dx + t dy$$

$$\therefore r = \frac{dp - s dy}{dx} \text{ and } t = \frac{dq - s dx}{dy}$$

Putting these values of r and t in (1), we get

$$\frac{dp - s dy}{dx} + (a+b)s + ab = \frac{dq - s dx}{dy} = xy$$

$$\text{or } \{dp dy + abd q dx - xy dx dy\} - s \{dy^2 - (a+b) dx dy + ab dx^2\} = 0$$

Monge's subsidiary equations are

$$dp dy + ab dq dx - xy dx dy = 0 \quad \dots(2)$$

$$dy^2 - (a+b) dx dy + ab dx^2 = 0 \quad \dots(3)$$

$$\text{and } dy^2 - (a+b) dx dy + ab dx^2 = 0$$

The equation (3) resolves in to the two equations $\dots(4)$

$$dy - b dx = 0 \quad \dots(5)$$

$$\text{and } dy - a dx = 0$$

$$\text{From (4), we get } y - bx = A$$

From (2) and (4), we have

$$dp + adq - xy dx = 0$$

or $dp + adq - x(A + bx) dx = 0$

Integrating, $p + aq - b \cdot \frac{1}{3}x^3 - A \cdot \frac{1}{2}x^2 = B$

$$\therefore p + aq - \frac{1}{3}bx^3 - \frac{1}{2}(y - bx)x^2 = f_1(y - bx)$$

or $p + aq + \frac{1}{6}bx^3 - \frac{1}{2}x^2y = f_1(y - bx)$

is an intermediate integral, f_1 being an arbitrary function.

Similarly from (2) and (5), we find another intermediate integrating

$$p + bq + \frac{1}{6}ax^3 - \frac{1}{2}x^2y = f_2(y - ax)$$

When f_2 is an arbitrary function

Solving (6) and (7) for p and q, we get

$$p = \frac{1}{2}x^2y - \frac{1}{6}(a+b)x^3 + \frac{1}{a-b}\{af_2(y - ax) - bf_1(y - bx)\}$$

and $q = \frac{1}{6}x^3 + \frac{1}{a-b}\{f_1(y - bx) - f_2(y - ax)\}$

Substituting these values of p and q, in

$$dz = p dx + q dy, \text{ we get}$$

$$dz = \frac{1}{2}x^2y dx + \frac{1}{6}x^3 dy - \frac{1}{6}(a+b)x^3 dx + \frac{1}{a-b}\{af_2(y - ax) - bf_1(y - bx)\} dx + \frac{1}{a-b}\{f_1(y - bx) - f_2(y - ax)\} dy$$

or

$$dz = \frac{1}{6}(3x^2y dx + x^3 dy) - \frac{1}{6}(a+b)x^3 dx + \frac{1}{(b-a)}[f_2(y - a)(dy - a dx) - f_1(y - bx)(dy - b dx)]$$

Integrating

$$z = \frac{1}{6}x^3y - \frac{1}{24}(a+b)x^4 + \frac{1}{b-a}\{\phi_2(y - ax) - \phi_1(y - bx)\}$$

or $z = \frac{1}{6}x^3y - \frac{1}{24}(a+b)x^4 + \psi_2(y - ax) + \psi_1(y - bx)$

Which is the complete integral of equation (1).

Example 43 : Solve $(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0$

Solution : We have

$$(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0 \quad \dots(1)$$

Also, we have

$$dp = r dx + s dy \text{ and } dq = s dx + t dy$$

$$\therefore r = \frac{dp - s dy}{dx} \text{ and } t = \frac{dq - s dx}{dy}$$

Putting these values of r and t in (1), we get

$$(1+q)^2 \frac{dp - s dy}{dx} - 2(1+p+q+pq)s + (1+p)^2 \frac{dq - s dx}{dy} = 0$$

or

$$\{(1+q)^2 d p dy + (1+p)^2 d q dx\} - s\{(1+q)^2 dy^2 + 2(1+p)(1+q) dx dy + (1+p)^2 dx^2\} = 0$$

Monge's subsidiary equations are

$$(1+q)^2 dp dy + (1+p)^2 dq dx = 0 \quad \dots(2)$$

and

$$(1+q)^2 dy^2 + 2(1+p)(1+q) dx dy + (1+p)^2 dx^2 = 0 \quad \dots(3)$$

The equation (3) gives $\{(1+q)dy + (1+p)dx\}^2 = 0$

$$\text{or } (1+q)dy + (1+p)dx = 0 \quad \dots(4)$$

$$\text{or } dx + dy + p dx + q dy = 0$$

$$\text{or } dx + dy + dz = 0 \quad [\because dz = p dx + q dy] \quad \dots(5)$$

Integrating, $x + y + z = a$

From (2) and (4), we get

$$(1+q)dp - (1+p)dq = 0$$

$$\text{or } \frac{dp}{1+p} = \frac{dq}{1+q}$$

Integrating, we get $(1+p) = (1+q)b$

From (5) and (6), we have $\dots(6)$

$$(1+p) = (1+q)f(x+y+z) \quad \dots(7)$$

Which is an intermediate integral, f being an arbitrary function.

The differential equation obtained in the above intermediate integral (7) can be written as

$$p - qf(x+y+z) = -1 + f(x+y+z)$$

Lagrange's auxiliary equations for this differential equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+y+z)} = \frac{dz}{-1+f(x+y+z)}$$

$$\text{Each function} = \frac{dx + dy + dz}{0}$$

$$\therefore dx + dy + dz = 0 \Rightarrow x + y + z = A$$

From the first two members, we get

$$\therefore dy = -f(A)dx$$

$$\therefore y = -xf(A) + B$$

$$\text{or } y + xf(x+y+z) = \phi(x+y+z)$$

Which is complete integral of equation (1).

EXERCISE 2.11 :

Solve

1. $r + (a+b)s + abt = xy$
2. $y^2r - 2ys + t = p + 6y$
3. $xy(t-r) + (x^2 - y^2)(s-2) = py - qx$
4. $(1+q)^2r - 2(1+p+q+pq)s + (1+p)^2t = 0$
5. $t - r \sec^4 y = 2q \tan y$
6. $(q+1)s = (p+1)t$
7. $(r-s)y + (s-t)x + q - p = 0$

ANSWER 2.11 :

- (1) $z = \frac{1}{6}x^3y - (a+b)\frac{x^4}{24} + F_1(y-ax) + F_2(y-bx)$
- (2) $z = y^3 - yF_1(y^2+2x) + F_2(y^2+2x)$
- (3) $z = xy + F_1(x^2+y^2) + F_2\left(\frac{x}{y}\right)$
- (4) $z = F_1(x+y+z) + xF_2(x+y+z)$
- (5) $z = F_1(x - \tan y) + F_2(x + \tan y)$
- (6) $z = f_1(x) + f_2(x+y+z)$
- (7) $z = f_1(x+y) + f_2(x^2-y^2)$

MULTIPLE CHOICE QUESTIONS

1. If $pq=1$ is partial differential equation then solution is
 (a) $z = ax - \frac{1}{a}y + c$ (b) $z = ax + \frac{1}{a}y + c$
 (c) $z = ax + by$ (d) None of these
02. If $p^2 - q^2 = 1$ is partial differential equation then solution is
 (a) $z = ax + \sqrt{(a^2 - 1)}y + c$ (b) $z = ax + (a^2 - 1)y + c$
 (c) $z = ax - (\sqrt{a^2 - 1})y + c$ (d) $z = ax - (a^2 - 1)y + c$
03. Complementary function of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \sigma \frac{\partial^2 z}{\partial y^2} = x + y$ is
 (a) $c.f. = f_1(y - 3x) + f_2(y + 2x)$ (b) $c.f. = f_1(y + 3x) + f_2(y + 2x)$
 (c) $c.f. = f_1(y - 3x) + f_2(y - 2x)$ (d) $c.f. = f_1(y + x) + f_2(y - x)$
04. Particular integral if the $(D^2 + DD' - \sigma D'^2)z = x^2 \sin(x + y)$ is
 (a) $P.I. = \frac{1}{D^2 + DD' - 6D'^2} x^2 \sin(x + y)$ (b) $P.I. = \frac{1}{D^2 + DD' - 6D'^2} x^2$
 (c) $P.I. = \frac{1}{D^2 - DD' - 6D'^2} x^2 \sin(x + y)$ (d) All the above
05. Which of the following Monge's equation
 (a) $R dp dy + T dq dx - V dx dy = 0$ (b) $R dy^2 - S dx dy + T dx^2 = 0$
 (c) Both a and b (d) $R dy^2 + S dx dy = 0$

06. Solution of the equation $(r - x)y + (s - t)x + q - p = 0$ is
 (a) $z = f_1(x + y) = f_2(x^2 - y^2)$ (b) $z = f_1(x + y) + f_2(x^2 + y^2)$
 (c) $z = f_1(x - y) + f_2(x + y)$ (d) All of these
07. Solution of $q + xp = p^2$ is,
 (a) $z = axe^{-y} - \frac{1}{2}a^2 e^{-2y} + b$ (b) $z = ay e^{-y} - \frac{1}{2}a^2 e^{-2y} + b$
 (c) $z = axe^y + \frac{1}{2}a^2 e^{2y} + b$ (d) All of these
08. Solution of $\sqrt{p} + \sqrt{q} = x + y$ is,
 (a) $3z = (x + a)^3 + (y - a)^3 + b$ (b) $3z = (x + a)^2 + (y - a)^2 + b$
 (c) $3z = (x + a)^3 - (y - a)^3 - b$ (d) All of these
09. Solution of $Z = p^2 + q^2$ is
 (a) $4z(1 + a^2) = (x + ay + b)^2$ (b) $4z(1 + a^2) = (x - ay - b)^2$
 (c) $4z(1 - a^2) = (x + ay - b)^2$ (d) None
10. Solution of $p^2 + q^2 = 1$ is,
 (a) $z = ax + \sqrt{(1 - a^2)}y + c$ (b) $z = ax + (1 - a^2)y + c$
 (c) $z = x + (1 - a^2)y + c$ (d) None
11. The equation $a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$ choose the following correct option
 (a) $a_0 m^2 + a_1 m + a_2 = 0$ (b) $a_0 m + a_1 m^2 + a_2 = 0$
 (c) $a_0 m^2 + a_1 + a_2 = 0$ (d) $a_0 m^2 + a_1 m - a_2 = 0$

12. The complementary function of $(D^2 - 4DD' + 4D'^2)z = 0$

(a) $z = f_1(y+2x) + xf_2(y+2x)$

(b) $z = f_1(y+2x) + f_2(y+2x)$

(c) $z = f_1(y-2x) + f_2(y-2x)$

(d) $z = f_1(y-2x) + xf_2(y-2x)$

13. The auxiliary equation of $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ is

(a) $m^2 - 2m + 2 = 0$

(b) $m^2 - 2m + 1 = 0$

(c) $m^2 + 2m + 1 = 0$

(d) None of these

14. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is

(a) $z = f_1(y) + f_2(y+x) + f_3(y+2x)$

(b) $z = f_1(y) + f_2(y+x)$

(c) $z = f_1(y) + f_2(y+x) + f_3(y-2x)$

(d) $z = f_1(y) + f_2(y-x) + f_3(y+2x)$

15. The auxiliary equation of $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ is

(a) $2m^2 + 5m + 2 = 0$

(b) $2m^2 - 5m - 2 = 0$

(c) $2m^2 + m + 2 = 0$

(d) All the above

16. Particular integral of $(D^3 - D^2D' + 4D'^3)z = e^{x+2y}$ is...

(a) $\frac{e^{x+2y}}{27}$

(b) $\frac{e^{x-2y}}{2}$

(c) $\frac{e^{x+y}}{3}$

(d) $\frac{e^{x-2y}}{27}$

17. Particular integral of $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{(3x-2y)}$ is...

(a) $\frac{e^{3x+2y}}{63}$

(b) $\frac{e^{3x-2y}}{63}$

(c) $\frac{e^{3x-2y}}{60}$

(d) $\frac{e^{3x+2y}}{36}$

18. Symbolic form of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - \sin x \cos 2y$ is

(a) $(D^2 + DD')z = \sin x \cos 2y$

(b) $(D^2 - DD')z = \sin x \cos 2y$

(c) $(D^2 - DD')z = -\sin x \cos 2y$

(d) None of the above

19. The partial differential equation form $z = (a+x)^2 + y$ is

(a) $z = \frac{1}{4} \left(\frac{\partial z}{\partial x} \right)^2 + y$

(b) $z = \frac{1}{4} \left(\frac{\partial z}{\partial y} \right)^2 + y$

(c) $z = \left(\frac{\partial z}{\partial x} \right)^2 + y$

(d) $z = \left(\frac{\partial z}{\partial y} \right)^2 + y$

20. The solution of $xp + yq = z$ is,

(a) $f(x, y) = 0$

(b) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

(c) $f(xy, yz) = 0$

(d) $f(x^2, y^2) = 0$

21. The solution $p + q = z$ is

(a) $f(x+y, y+\log z) = 0$

(b) $f(xy, y \log z) = 0$

(c) $f(x-y, y-\log z) = 0$

(d) None of these

22. The solution of $(y-z)p + (z-x)q = x-y$ is

(a) $f(x+y+z) = xyz$

(b) $f(x^2 + y^2 + z^2) = xyz$

(c) $f(x^2 + y^2 + z^2, x^2 y^2 z^2) = 0$

(d) $f(x+y+z) = x^2 + y^2 + z^2$



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23. The solution of $\frac{\partial^2 z}{\partial x^3} = 0$ is,

(a) $z = f_1(y) + xf_2(y) + x^2 f_3(y)$

(b) $z = (1 + x + x^2)f(y)$

(c) $z = f_1(x) + yf_2(x) + y^2 f_3(x)$

(d) $z = (1 + y + y^2)f(x)$

24. The solution of $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ is

(a) $z = f_1(y+x) + f_1(y-x)$

(b) $z = f_1(y+x) + f_2(y-x)$

(c) $z = f_2(y+x) + f_2(y-x)$

(d) $z = f(x^2 - y^2)$

25. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is,

(a) xe^{x+2y}

(b) $\frac{1}{2}e^{x+2y}$

(c) $-\frac{x}{2}e^{x+2y}$

(d) $\frac{x^2}{2}e^{x+2y}$

26. Particular integral of $(D^2 - D'^2)z = \cos(x+y)$ is

(a) $\frac{x}{2}\cos(x+y)$

(b) $x\sin(x+y)$

(c) $x\cos(x+y)$

(d) $\frac{x}{2}\sin(x+y)$

27. The complete integral of $p = e^q$ is,

(a) $z = ax + (Ina)y + c$

(b) $z = ax - (Ina)y + c$

(c) $z = ay + (Ina)y + c$

(d) $z = ax - (In)y + c$

28. The complete integral of $p^2 q^3 = 1$ is,

(a) $z = ax + a^{-\frac{2}{3}}y + c$

(b) $z = ax + a^{\frac{2}{3}}y + c$

(c) $z = ax - a^{\frac{2}{3}}y + c$

(d) $z = ax - a^{-\frac{2}{3}}y + c$

29. The solution of $r + 6s + 9t = 0$ is,

(a) $z = f_1(y-3x) - xf_2(y-3x)$

(b) $z = f_1(y+3x) - xf_2(y-3x)$

(c) $z = f_1(y-3x) + xf_2(y-3x)$

(d) $z = f_1(y-3x) - xf_2(y+3x)$

ANSWER KEY (MULTIPLE CHOICE QUESTIONS)

01.(b)	02.(a)	03.(a)	04.(a)	05.(c)	06.(a)
07.(a)	08.(a)	09.(a)	10.(a)	11.(a)	12.(a)
13.(b)	14.(a)	15.(a)	16.(a)	17.(a)	18.(b)
19.(a)	20.(b)	21.(c)	22.(d)	23.(a)	24.(b)
25.(c)	26.(d)	27.(a)	28.(a)	29.(c)	

PARTIAL DIFFERENTIAL EQUATIONS IN PRACTICAL PROBLEMS

3.1 INTRODUCTION :-

The following types of equations are generally used

- (i) One-dimensional heat flow : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- (iii) Two-dimensional flow : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- (iii) Wave equation : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
- (iv) Radio equations : $\frac{-\partial V}{\partial x} = L \frac{\partial I}{\partial t}, -\frac{\partial I}{\partial x} = C \frac{\partial V}{\partial t}$

3.2 METHOD OF SEPARATION OF VARIABLES

The following problems solve by using method of separation variables.

Example 1 : solve $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6e^{-3x}$

Solution :

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$$

Let

$$u = X(x).T(t) \quad \dots(1)$$

where X is a function of x only and T is a function of t only.

Putting the value of u in (1) we get

$$\dots(2)$$

$$\frac{\partial(X.T)}{\partial x} = 2 \frac{\partial}{\partial t}(X.T) + X.T$$

$$T \frac{dX}{dx} = 2X \frac{dT}{dt} + X.T \text{ or } T.X' = 2X.T' + X.T \text{ or } T \frac{X'}{X} = 2 \frac{T'}{T} + 1 = c \text{ (say)}$$

$$(a) \quad \frac{X'}{X} = c \text{ or } \frac{1}{X} \frac{dX}{dx} = C \text{ or } \frac{dX}{X} = c dx$$

$$\text{On integrating } \log X = cx + \log a \text{ or } \log \frac{X}{a} = cx \text{ or } \frac{X}{a} = e^{cx} \text{ or } X = ae^{cx}$$

$$(b) \quad \frac{2T'}{T} + 1 = c \text{ or } \frac{T'}{T} = \frac{1}{2}(c-1) \text{ or } \frac{1}{T} \frac{dT}{dt} = \frac{1}{2}(c-1) \text{ or } \frac{dT}{T} = \frac{1}{2}(c-1)dt$$

$$\text{On integration } \log T = \frac{1}{2}(c-1)t + \log b \text{ or } \log \frac{T}{b} = \frac{1}{2}(c-1)t$$

$$\text{or } \frac{T}{b} = e^{\frac{1}{2}(c-1)t} \text{ or } T = be^{\frac{1}{2}(c-1)t}$$

Putting the value of X and T in (2) we have

$$u = ae^{cx} be^{\frac{1}{2}(c-1)t} \quad \dots(3)$$

or

$$u = ab e^{cx + \frac{1}{2}(c-1)t}$$

or

$$u(x, 0) = ab e^{cx}$$

But

$$u(x, 0) = 6e^{-3x}$$

i.e.

$$ab e^{cx} = 6e^{-3x} \text{ or } ab = 6 \text{ and } c = -3$$

Putting the value of ab and c in (3), we have

$$u = 6e^{-3x + \frac{1}{2}(-3-1)t}$$

$$u = 6e^{-3x-2t}$$

Which is the required solution.

Example 2 : Use the method of separation of variables to solve the equation

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

given that $v=0$ what $t \rightarrow \infty$, as well as $v=0$, at $x=0$ and $x=l$

Solution :

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

...(1)

Let us assume that $v = XT$ where X is a function of x only and T that of t only

$$\frac{\partial v}{\partial t} = X \frac{dT}{dt}$$

$$\text{and } \frac{\partial^2 v}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting these value in (1) we get

$$X \frac{dT}{dt} = T \frac{d^2 X}{dx^2}$$

Let each side of (2) be equal to a constant $(-p^2)$

or

$$\frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = -p^2$$

...(2)

$$\frac{1}{T} \frac{dT}{dt} = -p^2$$

$$\text{or } \frac{dT}{dt} + p^2 T = 0$$

...(3)

and

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -p^2$$

$$\text{or } \frac{d^2 X}{dx^2} + p^2 X = 0$$

...(4)

Solving (3) and (4) we have

$$T = C_1 e^{-p^2 t}$$

$$X = C_2 \cos px + C_3 \sin px$$

$$v = C_1 e^{-p^2 t} (C_2 \cos px + C_3 \sin px) \quad \dots(5)$$

Putting $x=0, v=0$, in (5), we get

$$0 = C_1 e^{-p^2 t} C_2 \quad \therefore C_2 = 0, \text{ since } C_1 \neq 0$$

On putting the value of C_2 in (5), we get

$$v = C_1 e^{-p^2 t} C_3 \sin px \quad x=l, v=0 \text{ in (6) we get}$$

$$0 = C_1 e^{-p^2 t} C_3 \sin pl$$

Since C_3 cannot be zero

$$\therefore \sin pl = 0 = \sin n\pi \quad \therefore p = \frac{n\pi}{l}, n \text{ is any integer.}$$

On putting the value of p in (6) it becomes

$$v = C_1 C_3 e^{-\frac{n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

Hence

$$v = b_n e^{-\frac{n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

$$b_n = C_1 C_3$$

This equation satisfies the given condition for all integral values of n . Hence taking $n=1, 2, 3, \dots$ the most general solution is

$$v = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

Ans.

EXERCISE 3.1 :

Using the method of separation of variables, find the solution of the following equations

1.

$$2x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0$$

$$2. \frac{\partial u}{\partial x} + u = \frac{\partial u}{\partial t} \text{ if } u = 4e^{-3x} \text{ when } t=0$$

3. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ when $x = 0$
4. $4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, = 3e^{-x} - e^{-5x}$ at $t = 0$
5. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0, u(x, 0) = 4e^{-x}$
6. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u$
7. $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = 4x - \frac{1}{2}x^2$
8. $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ if $u(x, 0) = x(4 - x)$
9. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = 2x$ when $0 \leq x \leq \frac{1}{2}, = 2(l - x)$ when $\frac{1}{2} \leq x \leq l$
10. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \sin \pi x$
11. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ if $u(x, 0) = x^2(25 - x^2)$
12. $\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = u$, given that $u(x, 0) = 3e^{-5x} + 2e^{-3x}$
13. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$
14. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \frac{1}{2}x(1 - x)$
15. $16\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ if $u(x, 0) = x^2(5 - x)$
16. $\frac{\partial^2 y}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ if $u = 0, \frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$
17. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

ANSWER 3.1 :

1. $z = cx^{\frac{k}{2}}y^{\frac{k}{2}}$
2. $u = 4e^{-3x-2t}$

3. $u = e^{2x-5y}$
4. $u = 3e^{t-x} - e^{2t-5x}$
5. $u = 4e^{-x+3/2y}$
6. $u = ce^{x^3+y^2+k(x-y)}$
7. $u = \left(4x - \frac{x^2}{2}\right)e^{-p^2t}$
8. $u = x(4 - x)e^{\frac{p^2t}{2}}$
9. $u = 2xe^{-h^2t}$ for $0 \leq x \leq \frac{1}{2}, u = 2(l - x)e^{-h^2t}$ for $\frac{1}{2} \leq x \leq l$
10. $u \sin \pi x e^{xt}$
11. $u = x^2(25 - x^2)e^{xy}$
12. $u = e^{8x-5y}$
13. $z = c_1 e^{[1+\sqrt{1+p}]x+p^2y} + c_2 e^{[1-xqr-p](x+y)}$
14. $u = \frac{x}{2}(1 - x)\cos pt + c_2 \sin pt(c_3 \cos px + c_4 \sin px)$
15. $u = xh^2(5 - x)\cos pt + c_4 \sin pt\left(c_1 \cos \frac{px}{4} + c_2 \sin \frac{px}{4}\right)$
16. $u = \frac{1}{4}(e^{\sqrt{2}x} - e^{-\sqrt{2}x}) - \frac{1}{2}(e^{ix} - e^{-i})e^{-3x}$
17. $u = Ae^{1/2(t-1)}$

3.3 EQUATION OF VIBRATING STRING :

Let y be the displacement at the point $P(x, y)$ at any time. The wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

IMP

Example 3: Obtain the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

using the method of seapartion of variables.

Solution :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = X \frac{dT}{dt}$$

$$\text{and } \frac{\partial y}{\partial x} = T \frac{dX}{dx}$$

Since T and X are functions of a single variable only

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2}$$

$$\text{and } \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting these values in the given equation, we get

$$X = \frac{d^2 T}{dt^2} = C^2 T \frac{d^2 X}{dx^2}$$

By separating the variables, we get

$$\frac{\frac{d^2 T}{dt^2}}{C^2 T} = \frac{\frac{d^2 X}{dx^2}}{X} = k$$

(say)

(Each side is constant, since the variables x and y are independent)

$$\therefore \frac{d^2 T}{dt^2} = k C^2 T = 0$$

$$\text{and } \frac{d^2 X}{dx^2} - kX = 0$$

Auxiliary equations are

$$m^2 - kC^2 = 0 \text{ or } m = \pm C\sqrt{k} \text{ and } m^2 - k = 0 \text{ or } m = \pm\sqrt{k}$$

Case 1:

If $k > 0$

$$T = c_1 e^{C\sqrt{k}t} + C_2 e^{-C\sqrt{k}t}$$

$$X = c_3 e^{C\sqrt{k}x} + C_4 e^{-C\sqrt{k}x}$$

Case 2 :

If $k < 0$

$$T = c_5 \cos C\sqrt{k}t + C_6 \sin C\sqrt{k}t$$

$$X = C_7 \cos C\sqrt{k}x + C_8 \sin C\sqrt{k}x$$

Case 3 :

If $k = 0$

$$T = C_9 t + C_{10}$$

$$X = C_{11} x + C_{12}$$

These are three cases depending upon the particular problems. Here we are dealing with wave motion ($k < 0$)

$$y = TX$$

$$y = (C_5 \cos C\sqrt{k}t + C_6 \sin C\sqrt{k}t) \times (C_7 \cos \sqrt{k}x + C_8 \sin \sqrt{k}x) \quad \text{Ans.}$$

Example 4:

Find the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Such that $y = Po \cos pt$, (Po is a constant) when $x = l$ and $= 0$ when $x = 0$

...(1)

Solution :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Its solution is as given in Example 3.

$$y = (c_1 \cos \sqrt{k}t + c_2 \sin \sqrt{k}t)(c_3 \cos \sqrt{k}x + c_4 \sin \sqrt{k}x)$$

Put

$$y = 0, \text{ when } x = 0$$

$$0 = (c_1 \cos \sqrt{k}t + c_2 \sin \sqrt{k}t)c_3 \Rightarrow c_3 = 0$$

(2) is reduced to

$$y = (c_1 \cos c\sqrt{k}t + c_2 \sin c\sqrt{k}t) + c_4 \sin \sqrt{k}x$$

$$y = c_1 c_4 \cos c\sqrt{k}t \sin \sqrt{k}x + c_2 c_4 \sin c\sqrt{k}t \sin \sqrt{k}x$$

Put $y = P_0 \cos pt$ when $x = l$

$$P_0 \cos pt = c_1 c_4 \cos c\sqrt{k}t \sin \sqrt{k}l + c_2 c_4 \sin c\sqrt{k}t \sin \sqrt{k}l$$

Equating the coefficients of sin and cos on both sides

$$P_0 = c_1 c_4 \sin \sqrt{k}l, \Rightarrow c_1 c_4 = \frac{P_0}{\sin \sqrt{k}l}$$

$$0 = c_2 c_4 \sin \sqrt{k}l \Rightarrow c_2 = 0$$

And $p = c\sqrt{k} \Rightarrow \frac{p}{c} = \sqrt{k}$

(3) becomes $y = \frac{P_0}{\sin \sqrt{k}l} \cos pt \sin \frac{p}{c}x$

$$y = \frac{P_0}{\sin \frac{p}{c}l} \cos pt \sin \frac{p}{c}x$$

Ans.

Example 5 : A string is stretched and fastened to two points 1 apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at a time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = a \sin \left(\frac{\pi x}{l} \right) \cos \left(\frac{\pi x t}{l} \right)$$

Solution : The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$$

...(1)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad \dots(2)$$

and $y(l, t) = 0 \quad \dots(3)$

Since the initial transverse velocity of any point of the string is zero, therefore,

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots(4)$$

Also $y(x, 0) = a \sin \frac{\pi x}{l} \quad \dots(5)$

Now we have to solve (1), subject to the above boundary conditions. Since the vibration of the string is periodic, therefore, the solution of (1) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt) \quad \dots(6)$$

By (2) $y(0, t) = C_1(C_3 \cos Cpt + C_4 \sin Cpt) = 0$

For this to be true all time, $C_1 = 0$

Hence $y(x, t) = C_2 \sin px(C_3 \cos Cpt + C_4 \sin Cpt) \quad \dots(7)$

and $\frac{\partial y}{\partial t} = C_2 \sin px[C_3(-Cp \sin Cpt) + C_4(Cp \cos Cpt)]$

By (4) $\left(\frac{\partial y}{\partial t} \right)_{t=0} = C_2 \sin px(C_4 Cp) = 0$

Whence $C_2 C_4 Cp = 0$

If $C_2 = 0$, (7) will lead to the trivial solution $y(x, t) = 0$

$\therefore$ the only possibility is that $C_4 = 0$

Thus (7) becomes $y(x, t) = C_2 C_3 \sin px \cos Cpt \quad \dots(8)$

$$y(x, t) = C_2 C_3 \sin px \cos Cpt$$

If $x=l$ then $y=0$, $0 = C_2 C_3 \sin pl \cos Cpt$, for all t ,

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0 \quad \therefore pl = n\pi$

ie. $p = \frac{n\pi}{l}$, where n is an integer.

Hence (8) reduces to,

$$y(x, t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi Ct}{l} \quad \dots(9)$$

Finally imposing the last condition (5), we have

$$y(x, 0) = C_2 C_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$$

which will be satisfied by taking $C_2 C_3 = a$ and $n=1$

Hence the required solution is

$$y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi Ct}{l}$$

Proved.

Example 6 : The vibrations of an elastic string is governed by the partial differential equations

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string for $t > 0$.

Solution : $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow u = (c_1 \cos pt + c_2 \sin pt)(c_3 \cos px + c_4 \sin px) \quad \dots(1)$$

On putting, $x=0$, $u=0$ in (1), we get

$$0 = (c_1 \cos pt + c_2 \sin pt)c_3 \Rightarrow c_3 = 0$$

On putting $c_3 = 0$ in (1), it reduces

$$u = (c_1 \cos pt + c_2 \sin pt)c_4 \sin px \quad \dots(2)$$

On putting $x=\pi$ and $u=0$ in (2), we have

$$0 = (c_1 \cos pt + c_2 \sin pt)c_4 \sin p\pi$$

$$\sin p\pi = 0 = \sin n\pi \quad n=1, 2, 3, 4$$

$$p\pi = n\pi \text{ or } p=n$$

On substituting the value of p in (2) we get

$$u = (c_1 \cos nt + c_2 \sin nt)c_4 \sin nx \quad \dots(3)$$

On differentiating (3), w.r.t. 't' we get

$$\frac{du}{dt} = (-c_1 n \sin nt + c_2 n \cos nt)c_4 \sin nx \quad \dots(4)$$

On putting $\frac{du}{dt} = 0$, $t=0$ in (4) we have

$$0 = (c_1 n)(c_4 \sin nx) \Rightarrow c_2 = 0$$

On putting $c_2 = 0$, (3) becomes

$$u = (c_1 \cos nt)(c_4 \sin nx) \quad \dots(5)$$

$$u = c_1 c_4 \cos nt \sin nx$$

given $u(x, 0) = 2(\sin x + \sin 3x)$

On putting $t=0$ in (5) we have

$$u(x, 0) = c_1 c_4 \sin nx$$

$$2(\sin x + \sin 3x) = c_1 c_4 \sin nx$$

$$4 \sin 2x \cos x = c_1 c_4 \sin nx$$

$$c_1 c_4 = 4 \cos x$$

$$2 = n$$

On substituting the values of $c_1 c_4$ and $n=2$, (5) becomes

$$u(x, t) = 4 \cos x \cos 2t \sin 2x$$

Ans.

Example 7 : A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y(x, t)$

Solution : Let the equation to the vibrating string be

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Here the initial conditions are

$$y(0, t) = 0, y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t=0, y(x, 0) = y_0 \sin^3 \frac{\pi x}{l}$$

The solution (1) is of the form

$$y = (c_2 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$$

Now, $y(0, t) = 0$ gives

$$0 = c_1 (c_3 + \cos pct + c_4 \sin pct) \Rightarrow c_1 = 0$$

Hence (2) becomes

$$y = c_2 \sin px (c_3 \cos pct + c_4 \sin pct)$$

$y(l, t) = 0$ gives

$$0 = c_2 \sin pl (c_3 \cos pct + c_4 \sin pct)$$

...(1)

...(2)

...(3)

$$\sin pl = 0 = \sin n\pi \quad \text{or} \quad pl = n\pi \quad \text{or} \quad p = \frac{n\pi}{l} \quad \text{where} \quad n = 0, 1, 2, 3, \dots$$

On putting the value of p in (3) we get

$$y = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \quad \dots(4)$$

$$\text{Now,} \quad \frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l} \left(-\frac{n\pi x}{l} c_3 \sin \frac{n\pi c}{l} t + c_4 \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \right)$$

Since $\frac{\partial y}{\partial t} = 0$ when $t = 0$, we have

$$0 = c_2 \sin \frac{n\pi cx}{l} c_4 \frac{n\pi c}{l} \Rightarrow c_4 = 0$$

Now (4) reduces to

$$y = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad (b_n = c_2 c_3)$$

$$y = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t \quad \dots(5)$$

or

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t \quad \dots(6)$$

$$\text{But} \quad y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \frac{y_0}{4} \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) \quad (\text{given...})$$

$$\text{On putting } t = 0 \text{ in (5), we get } y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(7)$$

From (6) and (7), we have

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \frac{y_0}{4} \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right)$$

$$y = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} = \frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

Comparing the coefficients, we have

$$b_1 = \frac{3y_0}{4},$$

$$b_3 = -\frac{y_0}{4}$$

and all others b 's are zero

Hence (5) becomes

$$y = \frac{y_0}{4} \left(3 \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \cos \frac{3c\pi t}{l} \right)$$

Ans

Example 8 : Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

under the condition

$$u = 0 \text{ when } x = 0 \text{ and } x = \pi$$

$$\frac{\partial u}{\partial t} = 0 \text{ when } t = 0 \text{ and } u(x, 0) = x, 0 < x < \pi$$

Solution : The solution is of the form

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos a pt + c_4 \sin a pt)$$

$$\text{Since } u(0, t) = 0$$

$$0 = c_1 (c_3 \cos a pt + c_4 \sin a pt) \Rightarrow c_1 = 0$$

Then (1) becomes

$$u(x, t) = c_2 \sin px (c_3 \cos a pt + c_4 \sin a pt)$$

$$u(\pi, t) = 0$$

...(2)

$$0 = c_2 \sin p\pi (c_3 \cos a pt + c_4 \sin a pt) \Rightarrow \sin p\pi = 0 = \sin n\pi \text{ or } p = n$$

$$\text{Thus } u(x, t) = c_2 \sin nx (c_3 \cos an + c_4 \sin ant)$$

$$u(x, t) = \sin nx (b_1 \cos ant + b_2 \sin ant) \quad \dots(3)$$

$$\text{Now } \frac{\partial u}{\partial t} = \sin nx (-ab_1 n \sin ant + ab_2 n \cos ant)$$

$$\text{As } \frac{\partial u}{\partial t} = 0 \text{ when } t = 0 \text{ we have}$$

$$0 = \sin nx (ab_2 n) \Rightarrow b_2 = 0$$

$$u(x, t) = \sin nx (b_1 \cos ant) \text{ or } u(x, t) = b_n \sin nx \cos ant \quad \dots(4)$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin nx \cos ant \quad \dots(5)$$

On putting $u(x, 0) = x$, we have

$$x = \sum_{n=1}^{\infty} b_n \sin nx, \text{ where } b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= \frac{2}{\pi} \left[x - \left(-\frac{\cos nx}{n} \right) - (1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} \cos n\pi \right] = -\frac{2}{n} (-1)^n$$

Hence the required solution is

$$u(x, t) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \cos n at$$

Ans.

Example 9 : A string is stretched to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the

displacement of any point on the string at a distance of x from one end at time t .

Solution : The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

As the end points of the string are fixed for all time.,

$$y(0, t) = 0 \quad \dots(2)$$

and

$$y(l, t) = 0 \quad \dots(3)$$

since the initial transverse velocity of any point of the string is zero, therefore

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots(4)$$

and

$$y(x, 0) = k(lx - x^2) \quad \dots(5)$$

Solution of (1) is $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos c pt + c_4 \sin c pt)$

By (2) $y(0, t) = 0$... (6)

$$0 = c_1 (c_3 \cos c pt + c_4 \sin c p) \quad \therefore c_1 = 0$$

Hence (6) becomes $y = c_2 \sin px (c_3 \cos c pt + c_4 \sin c pt)$... (7)

$$\frac{\partial y}{\partial t} = c_2 \sin px (-c_3 c p \sin c pt + c_4 c p \cos c pt)$$

By (4) $\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$

$$0 = c_2 \sin px (c_4 c p) \Rightarrow \therefore c_4 = 0 \quad \text{since } c_2 \neq 0$$

Hence (7) is reduced to

$$y = c_2 \sin px (c_3 c p)$$

$$y = c_2 c_3 \sin px \cos c pt \quad \dots(8)$$

$$y(l, t) = 0$$

On putting $x = l$ in equation (8) we get

$$0 = c_2 c_3 \sin pl \cos c pt \quad \text{or} \quad 0 = \sin pl$$

$$\sin n\pi = \sin pl \quad \text{or} \quad pl = n\pi, p = \frac{n\pi}{l} \quad \text{where } n = 1, 2, 3, \dots$$

On putting $p = \frac{n\pi}{l}$, equation (8) becomes

$$(c_2 c_3 = b_n)$$

or

$$y = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t$$

We can have any number of solutions by taking different integral values of n and the complete solution will be the sum of these solutions. Thus,

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t \quad \dots(9)$$

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(10)$$

$$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

[using (5)]

Now it is clear that (10) represents the expansion of $f(x)$ in the form of a Fourier sine series and consequently

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$b_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx \quad \dots(11)$$

$$= \frac{2}{l} \left[(lx - x^2) \left(-\cos \frac{n\pi x}{l} \right) \frac{l}{n\pi} - (l-2x) \left(-\sin \frac{n\pi x}{l} \right) \frac{l^2}{n^2\pi^2} + (-2) \left(\cos \frac{n\pi x}{l} \right) \frac{l^3}{n^3\pi^3} \right]_0^l$$

$$= \frac{2}{l} \left[(-1)^{n+1} \frac{2l^3}{n^2\pi^3} + \frac{2l^3}{n^3\pi^3} \right] = \frac{8l^2}{n^3\pi^3} \text{ when } n \text{ is odd}$$

$$= 0 \text{ when } n \text{ is even}$$

Putting the value of b_n in (9) we get

Ans.

$$y = \sum \frac{8l^2}{n^3\pi^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \text{ when } n \text{ is odd.}$$

3.4 SOLUTION OF WAVE EQUATION BY D'ALMBERT'S METHOD :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

Let us introduce the two new independent variables $u = x + ct$, $v = x - ct$
So that y becomes a function of u and v

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} (1) + \frac{\partial y}{\partial v} (-1) = \frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right)$$

$$= \frac{\partial}{\partial u} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad \dots(2)$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial t} = \frac{\partial y}{\partial u} c + \frac{\partial y}{\partial v} (-c) = c \left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right) \left[\because \frac{\partial u}{\partial t} = c, \frac{\partial v}{\partial t} = -c \right]$$

$$\frac{\partial}{\partial t} = c \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) = c \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) c \left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right)$$

$$= c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \quad \dots(3)$$

Substituting the values, $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ from (2) and (3) in (1), we get

$$c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) = c^2 \left(\frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \text{ or } \frac{\partial^2 y}{\partial u \partial v} = 0 \quad \dots(4)$$

Integrating (4) w.r.t. v , we get $\frac{\partial y}{\partial u} = f(u)$

where $f(u)$ is constant in respect v . Again integrating (5) w.r.t. ' u ', we get

$$y = \int f(u) du + \psi(v)$$

where $\psi(v)$ is constant in respect of u

$$y = \phi(u) + \psi(v) \text{ where } \phi(u) = \int f(u) du \quad \dots(6)$$

$\Rightarrow$

$$y(x, t) = \phi(x + ct) + \psi(x - ct)$$

This is D'Almberts solution of wave equations (1)

To determine ϕ, ψ let us apply initial conditions $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t} = 0$
when $t = 0$.

Differentiating (6) w.r.t. ' t ', we get

$$\frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct) \quad \dots(7)$$

Putting $\frac{\partial y}{\partial t} = 0$ and $t = 0$ in (7) we get $0 = \phi'(x) - \psi'(x)$

or

$$\phi'(x) = \psi'(x) \text{ or } \phi(x) = \psi(x) + b$$

Again substituting $y = f(x)$ and $t = 0$ in (6) we get

$$f(x) = \phi(x) + \psi(x) \text{ or } f(x) = [\psi(x) + b] + \psi(x)$$

$$f(x) = 2\psi(x) + b$$

$$\text{so that } \psi(x) = \frac{1}{2}[f(x) - b] \text{ and } \phi(x) = \frac{1}{2}[f(x) + b]$$

On putting the values of $\phi(x+ct)$ and $\psi(x-ct)$ in (6), we get

$$y(x, t) = \frac{1}{2}[f(x+ct) + b] + \frac{1}{2}[f(x-ct) - b]$$

$$y(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Ans.

EXERCISE 3.2 :

1. Solve the wave equations $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u = 0$ when $x = 0$ and $x = \pi$
 $\frac{\partial u}{\partial t} = 0$ when $t = 0$ and $u(x, 0) = x$, $0 < x < \pi$.
2. Using the transform of $v = x + ct$ and $z = x - ct$, solve the following.
 $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\frac{\partial u}{\partial t}(x, 0) = 0$; $u(x = 0) = f(x)$.

3. A string of length l is initially at rest in equilibrium position and each of its points is given velocity

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$$

Find the displacement $y(x, t)$.

4. Find the solution of the equation of vibrating string of length l satisfying the initial conditions

$$y = f(x) \text{ when } t = 0, \text{ and } \frac{\partial y}{\partial t} = g(x) \text{ when } t = 0$$

It is assumed that the equation of a vibrating string is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

05. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t .
06. A tightly stretched string of length l fastened at both ends, is disturbed from the position of equilibrium by imparting to each of its points an initial velocity of magnitude $f(x)$. Show that the solution of the problem is

$$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} \left[\int_0^l f(x) \sin \frac{n\pi x}{l} dx \right] \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

07. A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = .03 \sin x - .04 \sin 3x$ then find the displacement $y(x, t)$ at any point of the string at any time t .

08. Find the solution of the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ which satisfies the conditions

$$u(0, t) = 0, u(l, t) = 0, u(x, 0) = \phi(x), u(x, 0) = 0$$

1. $u = \frac{2\sum(-1)^{n+1}}{n} \sin x \cos nct$
2. $u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$
3. $y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$
4. $y(x, t) = \sum_{n=1}^{\infty} \left(b_n \cos \frac{n\pi at}{l} + c_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$ where
 $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, c_n = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$
05. $y = \frac{8\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi ct}{l}$

3.5 ONE DIMENSIONAL HEAT FLOW :

Let the temperature of the bar at any time t at a point x distance from the origin be $u(x, t)$. Then the equation of one dimensional heat flow $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Example 10 : A rod of length l with insulated sides is initially at a uniform temperature u . Its ends are suddenly cooled at 0°C and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 \pi^2 n^2 t}{l^2}}$$

where b_n is determined from the equation

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

Let us assume that $u = XT$, where X is a function of x alone and T that of t alone

$$\frac{\partial u}{\partial t} = X \frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting these values in (1), we get $X = \frac{dT}{dt} = c^2 T \frac{d^2 X}{dx^2}$

$$\text{or} \quad \frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} \quad \dots(2)$$

Let each side be equal to a constant $(-p^2)$

$$\frac{1}{c^2 T} \frac{dT}{dt} = -p^2 \quad \text{or} \quad \frac{dT}{dt} + p^2 c^2 T = 0 \quad \dots(3)$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -p^2 \quad \text{or} \quad \frac{d^2 X}{dx^2} + p^2 X = 0 \quad \dots(4)$$

Solving (3) and (4) we have

$$T = c_1 e^{-p^2 c^2 t} \quad \text{and} \quad X = c_2 \cos px + c_3 \sin px \quad \dots(5)$$

$$u = c_1 e^{-p^2 c^2 t} (c_2 \cos px + c_3 \sin px)$$

Putting $x = 0, u = 0$ in (5),

$$\text{we get } = c_1 e^{-p^2 c^2 t} (c_2) \Rightarrow c_2 = 0, c_1 \neq 0 \quad (5) \text{ becomes } u = c_1 e^{-p^2 c^2 t} c_3 \sin px \quad \dots(6)$$

$$\text{Again putting } x=l, u=0 \text{ in (6) we get } \Rightarrow pl = c_1 e^{-p^2 c^2 t} c_3 \sin pl \Rightarrow \sin pl = 0 = \sin n\pi$$

$$\Rightarrow pl = n\pi \quad \text{or} \quad \therefore p = \frac{n\pi}{l}, n \text{ is any integer.}$$

Hence (6) becomes $u = c_1 c_3 e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l} = b_n e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$, $b_n = c_1 c_3$

This equation satisfies the given conditions for all integral values on n . Hence taking $n=1, 2, 3, \dots$ the most general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

By initial conditions $u = U_0$ when $t = 0$

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Proved

Example 11 : Find the solution of $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ for which $u(0, t) = u(l, t) = 0, u(x, 0) = \sin \frac{\pi x}{l}$ by method of variables separable.

Solution : $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$

In example 10 the given equation was

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

On comparing (1) and (2) we get $h^2 = \frac{1}{c^2}$

Thus solution of (1) is

$$u = (c_2 \cos px + c_3 \sin px) c_1 e^{\frac{-p^2 t}{h^2}}$$

On putting $x = 0, u = 0$ in (3) we get

$$0 = c_1 c_2 e^{\frac{-p^2 t}{h^2}}$$

$$c_1 \neq 0 \therefore c_2 = 0$$

(3) is reduced to

$$u = c_3 \sin pxc_1 e^{\frac{-p^2 t}{h^2}} \quad \dots(4)$$

On putting $x = l$ and $u = 0$ in (4), we get

$$0 = c_3 \sin pl c_1 e^{\frac{-p^2 t}{h^2}}$$

$$c_3 \neq 0, c_1 \neq 0$$

$$\left[\therefore \sin pl = 0 = \sin n\pi \text{ or } p = \frac{n\pi}{l} \right]$$

Now (4) is reduced to

$$u = c_1 c_3 \sin \frac{n\pi x}{l} e^{\frac{-n^2 \pi^2 t^2}{h^2 l^2}} \quad \dots(5)$$

On putting $t = 0, u = \sin \frac{\pi x}{l}$ in (5) we get

$$\sin \frac{\pi x}{l} = c_4 \sin \frac{n\pi x}{l}$$

$$[\text{put } c_1 c_3 = c_4]$$

This equation will be satisfied if

$$n = 1 \text{ and } c_4 = 1$$

On putting the values of c_4 and n in (5), we have

Ans.

$$u = \sin \frac{\pi x}{l} e^{\frac{-\pi^2 t}{h^2 l^2}}$$

Example 12 : The ends A and B of a rod 20 cm long have the temperatures at 30°C and at 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 80°C respectively. Find the temperature distribution in the rod at time t .

Solution : The initial temperature distribution in the rod is

$$u = 30 + \frac{50}{20}x$$

$$\text{ie. } u = 30 + \frac{5}{2}x$$

and the final distribution (i.e. in steady state) is

$$u = 40 + \frac{20}{20}x = 40x$$

To get u in the intermediate period, reckoning time from the instant when the end temperature were changed, we assumed

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod (i.e. temperature after sufficiently long time) and $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases

Thus

$$u_2(x) = 40 + x$$

Now $u_1(x, t)$ satisfies the one-dimensional heat-flow equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

hence u is of the form

$$u = 40 + x + \sum (a_k \cos kx + b_k \sin kx) e^{-c^2 k^2 t}$$

Since

$$u = 40^0 \text{ when } x = 0 \text{ and } u = 60^0 \text{ when } x = 20, \text{ we get}$$

$$a_k = 0, k = \frac{n\pi}{20}$$

Hence

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-c^2 \left(\frac{n\pi}{20}\right)^2 t} \quad \dots(1)$$

Using the initial condition i.e.,

$$u = 30 + \frac{5}{2}x \text{ when } t = 0, \text{ we get}$$

$$30 + \frac{5}{2}x = 40 + x + \sum b_n \sin \frac{n\pi x}{20} \quad \text{or} \quad \frac{3}{2}x - 10 = \sum b_n \sin \frac{n\pi x}{20}$$

Hence

$$\begin{aligned} b_n &= \frac{2}{20} \int_0^{20} \left(\frac{3}{2}x - 10 \right) \sin \frac{n\pi x}{20} dx \\ &= \frac{1}{10} \left[\left(\frac{3x}{2} - 10 \right) \left(-\frac{20}{n\pi} \cos \frac{n\pi x}{20} \right) - \frac{3}{2} \left(\frac{-400}{n^2 \pi^2} \sin \frac{n\pi x}{20} \right) \right]_0^{20} \\ &= \frac{1}{10} \left[-20 \left(\frac{20}{n\pi} \right) (-1)^n - (-10) \left(\frac{20}{n\pi} \right) \right] = \frac{-20}{n\pi} [2(-1)^n + 1] \end{aligned}$$

Putting this value of b_n in (1) we get

$$u = 40 + x - \frac{20}{\pi} \sum \left[\frac{2(-1)^n}{n} \right] \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t} \quad \text{Ans.}$$

EXERCISE 3.3 :

1. Solve the following boundary value problem which arises in the heat conduction in a rod

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad u(0, t) = u(l, t) = 0$$

$$u(x, 0) = 100 \frac{x}{l}$$

2.

Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary condition $u(0, t) = 0, u(l, t) = 0 (t > 0)$ and the initial condition

3. $u(x, 0) = x, l$ being the length of the bar.

Solve the non-homogenous heat conduction equation $u = a^2 u_{xx} + \sin 3\pi x$ subject to the following conditions

$$u(x, 0) = \sin 2\pi x, u(0, t) = u(l, t) = 0$$

4. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ given that (i) $u = 0$ when $x = 0$ and $x = l$ for all t

(ii) $u = 3 \sin \frac{\pi x}{l}$, when $t = 0$ for all $0 < x < l$

5. (a) Find by the method of separation of variables the solution $U(x, t)$ of the boundary value problem

$$\frac{\partial U}{\partial t} = 3 \frac{\partial^2 U}{\partial x^2}$$

$$t > 0, 0 < x < 2$$

$$U(0, t) = 0$$

$$U(2, t) = 0$$

$$U(x, 0) = x$$

$$0 < x < 2$$

- (b) The ends A and B of a rod 30 cm long have their temperature kept at 20°C and 80°C respectively until steady-state conditions prevail. The temperature of the end B is suddenly reduced to 60°C and kept so while at the end A temperature is raised to 40°C . Find temperature distribution in the rod of time t .

6. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar, given that $u = u_0 \sin nt$ when for all values of t and $u = 0$ when x is very large.

Without radiation show that if $u = Ae^{-gx} \sin(nt - gx)$, where A , g and n are positive

constants then $g = \sqrt{\frac{n}{2\mu}}$

7. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively and steady state condition prevails. If B is suddenly reduced to 0°C and maintained at 0°C , find its temperature at a distance x from A at time t , solve the equation of heat

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

by the method of separation of variables and obtain the solution.

8. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ under the condition

$$u'(0, t) = 0 \quad t > 0$$

$$u'(\pi, t) = 0$$

$$u(x, 0) = x^2, \quad 0 < x < \pi$$

9. A rod of length l has its lateral surface insulated and is so thin that heat flow in the rod can be regarded as one dimensional. Initially the rod is at the temperature 100 throughout. At $t = 0$ the temperature at the left end of the rod is suddenly reduced to 50 and maintained thereafter at this value while the right end is maintained at 100. Let $u(x, t)$ be the temperature at point x in the rod is any subsequent time t .

- (i) Write down the appropriate partial differential equation or $u(x, t)$ with initial and boundary conditions.

- (ii) Solve the different equation (i) above using method of separation of variables and show that

$$u(x, t) = 50 + \frac{50x}{l} + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \exp \frac{-n^2 \pi^2 t}{a^2 l^2}$$

10. where a^2 is the constant involved in the partial differential equation. A uniform rod of length a whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature $\theta(x, t)$

11. Solve $\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}$ under the conditions

- (i) $U \neq \alpha$ if $t \rightarrow \alpha$ (ii) $U(0, t) = U(\pi, t) = 0$ (iii) $U(x, 0) = \pi x - x^2$

12. The temperature distribution in a bar length π , which is perfectly insulated at the ends $x = 0$ and $x = \pi$ is governed by the partial differential equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$. Assuming

the initial temperature as $u(x, 0) = f(x) = \cos 2x$ find the temperature distribution at any instant of time..

ANSWER 3.3 :

1. Ans. $u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$
2. Ans. $y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l} e^{-\frac{n^2 c^2 \pi^2 t}{l^2}}$
4. Ans. $u = 3 \sin \frac{\pi x}{l} e^{-\frac{a^2 \pi^2 t}{l^2}}$
5. (a) Ans. $U = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{2}}{\sin \frac{n\pi}{2}} e^{-\frac{3n^2 \pi^2 t}{4}}$
7. Ans. $u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$
8. Ans. $u(x, t) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx e^{-c^2 \pi^2 n^2 t}$
10. Ans. $\theta(x, t) = \frac{4\theta_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n+1)\pi x}{2a} \frac{1}{2n+1} e^{-\frac{(2n+1)^2 \pi^2 c^2 t}{4a^2}}$
11. Ans. $u = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} e^{-a^2(2n-1)^2 t}$

3.6 TWO DIMENSIONAL HEAT FLOW :

Let the heat flow in a metal of uniform thickness, in the parallel direction to length and width of the plane. There is no heat flow along the normal to the plane of the rectangle. Let $u(x, y)$ be the temperature at any point (x, y) of the plate at time t is given by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

In the steady state, u does not change with t .

$$\frac{\partial u}{\partial t} = 0$$

$$(1) \text{ becomes } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is called Laplace equation in two dimensions.

Example 13 : Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0$$

and

$$u(x, a) = \sin \frac{n\pi x}{l}$$

...(1)

Solution :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

...(2)

Let

$$u = X(x), Y(y)$$

Putting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in (1) we have

$$X''Y + XY'' = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} = -p^2 \quad (\text{say}) \quad \dots(3)$$

$\therefore$

$$X'' = -p^2 X \quad \text{or} \quad X'' + p^2 X = 0 \quad \dots(4)$$

and

$$Y'' = p^2 Y \quad \text{or} \quad Y'' - p^2 Y = 0$$

A.E. of (3) is $m^2 + p^2 = 0$ or $m = \pm ip$

$\therefore$

$$X = c_1 \cos px + c_2 \sin px$$

A.E. of (4) is $m^2 - p^2 = 0$ or $m = \pm P$

$\therefore$

$$Y = c_3 e^{Py} + c_4 e^{-Py}$$

Putting the values of X and Y in (2) we get

$$u = (c_1 \cos Px + c_2 \sin Px)(c_3 e^{Py} + c_4 e^{-Py}) \quad \dots(5)$$

Putting $x=0, u=0$, in (5) we have

$$0 = c_1 (c_3 e^{Py} + c_4 e^{-Py})$$

$$c_1 = 0$$

(5) is reduced to

$$u = c_2 \sin Px (c_3 e^{Py} + c_4 e^{-Py}) \quad \dots(6)$$

On putting $x=l, u=0$, we have

$$0 = c_2 \sin pl (c_3 e^{Py} + c_4 e^{-Py})$$

$$c_2 \neq 0$$

$$\therefore \sin pl = 0 = \sin n\pi$$

or

$$pl = n\pi \quad \text{or} \quad P = \frac{n\pi}{l}$$

Now (6) becomes

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right) \quad \dots(7)$$

On putting $u=0$ and $y=0$ in (7) we have

$$0 = c_2 \sin \frac{n\pi x}{l} (c_3 + c_4)$$

$$c_3 + c_4 = 0 \quad \text{or} \quad c_3 = -c_4$$

$$(7) \text{ becomes } u = c_2 c_3 \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right) \quad \dots(8)$$

On putting $y=a$ and $u = \sin \frac{n\pi x}{l}$ in (8), we get

$$\sin \frac{n\pi x}{l} = c_2 c_3 \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}} \right) \text{ i.e. } c_2 c_3 = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}}$$

Putting this value in (8) we have

$$u = \sin \frac{n\pi x}{l} \frac{e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}} \quad \text{or} \quad u = \sin \frac{n\pi x}{l} \frac{\sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}}$$

Ans

Example 14 : A rectangular plate with insulated surfaces is 10 cm wide and as long compared to its width that it may be considered infinite in length without introducing an applicable error. If the temperature along the short edge $y=0$ is given by

$$u(x, 0) = 20x \quad 0 < x \leq 5$$

$$= 20(10 - x), \quad 5 < x < 10$$

While the two long edges $x=0$ and $x=10$ as well as the other short edges kept at 0°C . Find the steady state temperature at any point (x, y) of the plate.

Solution : In the steady state, the temperature $u(x, y)$ at any point $p(x, y)$ satisfy the equation

$$\dots(1)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\dots(2)$$

The boundary conditions are,

$$u(0, y) = 0 \quad \text{for all values of } y$$

$$\dots(3)$$

$$u(10, y) = 0 \quad \text{for all values of } y$$

$$u(x, \infty) = 0 \text{ for all values of } x \quad \dots(4)$$

$$u(x, 0) = 20x \quad 0 < x \leq 5$$

$$= 20(10 - x), 5 < x < 10 \quad \dots(5)$$

Now three possible solutions of (1) we

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad \dots(6)$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \quad \dots(7)$$

$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \quad \dots(8)$$

Of these we have to choose that solution which is consistent with the physical nature of the problem. The solution (6) and (8) cannot satisfy the condition (2), (3) and (4). Thus only possible solution is (7) i.e. of the form.

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \dots(9)$$

By (2) $u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all values of y

$$C_1 = 0$$

(9) reduces to $u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py})$

$$\dots(10)$$

By (3) $u(10, y) = C_2 \sin 10p (C_3 e^{py} + C_4 e^{-py}) = 0$ $C_2 \neq 0$

$$\sin 10p = 0$$

Also to satisfy the condition (4) i.e. $u = 0$ as $y \rightarrow \infty$

$$C_3 = 0$$

Hence (10) takes the form $u(x, y) = C_2 C_4 \sin px e^{-py}$

or $u(x, y) = b_n \sin px e^{-py}$

$$\text{where } b_n = C_2 C_4$$

$\therefore$ The most general solution that satisfies (2), (3), & (4) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin px e^{-py}$$

Putting $y = 0$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin px \quad \text{where } p = \frac{n\pi}{10}$$

This requires the expansion of u in Fourier series in the interval $x = 0$ and $x = 5$ and from $x = 5$ to $x = 10$

$$b_n = \frac{2}{10} \int_0^5 20x \sin px dx + \frac{2}{10} \int_5^{10} 20(10 - x) \sin px dx$$

$$b_n = 4 \int_0^5 x \sin px dx + 4 \int_5^{10} (10 - x) \sin px dx$$

$$= 4 \left[x \left(\frac{-\cos px}{p} \right) - (-1) \left(\frac{-\sin px}{p^2} \right) \right]_0^5 + 4 \left[(10 - x) \left(\frac{-\cos px}{p} \right) - (-1) \left(\frac{-\sin px}{p^2} \right) \right]_5^{10}$$

$$= 4 \left[\frac{-5 \cos 5P}{p} + \frac{\sin 5P}{p^2} \right] + 4 \left[0 - \frac{\sin 10P}{p^2} + \frac{5 \cos 5P}{p} + \frac{\sin 5P}{p^2} \right]$$

$$= 4 \left[\frac{2 \sin 5P}{p} - \frac{\sin 5P}{p^2} \right] \quad \left(P = \frac{n\pi}{10} \right)$$

$$= 4 \left[\frac{2 \sin 5 \cdot \frac{n\pi}{10}}{\frac{n^2 \pi^2}{100}} - \frac{\sin 10 \cdot \frac{n\pi}{10}}{\frac{n^2 \pi^2}{100}} \right] = \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{400}{n^2 \pi^2} \sin n\pi$$

$$= \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2} = 0 \text{ if } n \text{ is even. } = \pm \frac{800}{n^2 \pi^2} \text{ if } n \text{ is odd. or } b_n = \frac{(-1)^{n+1} 800}{(2n-1)^2 \pi^2}$$

On putting the value of b_n in (5) the temperature at any point (x, y) is given by

$$u(x, y) = \frac{800}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} e^{-\frac{(2n-1)\pi y}{10}} \quad \text{Ans.}$$

EXERCISE 3.4 :

1. Find by the method of separation of variables, a particular solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

that tends to zero as x tends to infinity and is equal to $2 \cos y$ when $x = 0$.

2. Solve the equation $u_{xx} + u_{yy} = 0$

$$u(0, y) = u(\pi, y) = 0 \text{ for all } y$$

$$u(x, 0) = k \quad 0 < x < \pi$$

$$\lim_{y \rightarrow \infty} u(x, y) = 0 \quad 0 < x < \pi$$

3. Solve the boundary value problem subject to the given conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$0 \leq x \leq 10$$

$$0 \leq y < \infty$$

$$u(0, y) = 0$$

$$\text{for all } y$$

$$u(0, y) = 0$$

for all

$$u(x, \infty) = 0$$

$$5 \leq x \leq 10$$

$$\text{in } 0 < x < 5$$

$$u(x, 0) = 20(10 - x)$$

4. Find the solution of Laplace's equation $\Delta^2 \psi = 0$ in cartesian coordinates in the region $0 \leq x \leq a, 0 \leq y \leq b_0$ to satisfying the conditions $y = 0$ on $x = 0, x = a, y = b$ and $\psi = x(a - x), 0 < x < a$
5. An infinitely long uniform plate is bounded by two parallel edges and end at right angles to them. The breadth is p . This end is maintained at a temperature u_0 at all points and

other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

6. The points of trisection of a rightly stretched string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

7. The steady state temperature distribution in a thin plate bounded by the line $x = 0, x = a, y = 0$ and $y = \infty$ is governed by the partial differential equation.

Obtain the steady state temperature distribution under the conditions

$$\begin{aligned} u(x, 0) = 0 & \quad u(a, y) = 0 & \quad u(x, \infty) = 0 \\ u(x, 0) = x, & \quad 0 \leq x \leq a/2 \\ & \quad a/2 \leq x \leq a \\ & \quad = a - x \end{aligned}$$

ANSWER 3.4 :

1.

$$u = 2e^{-1} \cos y$$

2.

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx e^{-ny}, \sum_{n=1}^{\infty} b_n \sin nx$$

3.

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} e^{-\frac{(2n-1)\pi y}{10}}$$

4.

$$\psi = \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^3} \frac{\sin \frac{\pi x}{a} \sinh \frac{(2n+1)\pi y}{a}}{\sinh \frac{(2n+1)\pi b}{a}}$$

5.

$$u(x, y) = \frac{4u_0}{\pi} \left[e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \dots \right]$$

3.7 LAPLACE EQUATION IN POLAR CO-ORDINATES :

Example 15 : Solve $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

by the method of separation of variables.

Solution : $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ or $r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2} = 0$... (1)

Let $u = R(r) \cdot T(\theta)$

$$\frac{\partial u}{\partial r} = \frac{dR}{dr} T(\theta)$$

$$\text{and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 R}{dr^2} T(\theta)$$

$$\frac{\partial u}{\partial \theta} = R(r) \frac{dT}{d\theta}$$

$$\text{and } \frac{\partial^2 u}{\partial \theta^2} = R(r) \frac{d^2 T}{d\theta^2}$$

Putting the values of $\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial u}{\partial r}$ and $\frac{\partial^2 u}{\partial \theta^2}$ in (1) we get

$$r^2 \cdot \frac{d^2 R}{dr^2} T(\theta) + r \frac{dR}{dr} T(\theta) + R(r) \frac{d^2 T}{d\theta^2} = 0$$

$$\left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) T + R \frac{d^2 T}{d\theta^2} = 0$$

$$\frac{r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr}}{R} = -\frac{1}{T} \frac{d^2 T}{d\theta^2} = h$$

(say)

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - hR = 0$$

Put $r = e^z$

$$(D(D-1) + D - h)R = 0$$

$$\frac{d^2 T}{d\theta^2} + hT = 0$$

$$(D^2 + h)T = 0$$

$$D^2 + h = 0 \text{ or } D = \pm i\sqrt{h}$$

$$D^2 - h = 0 \text{ or } D = \pm \sqrt{h}$$

$$R = c_1 e^{\sqrt{h}z} + c_2 e^{-\sqrt{h}z}$$

$$R = c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}}$$

$$u = (c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}}) [c_3 \cos(\sqrt{h}\theta) + c_4 \sin(\sqrt{h}\theta)] \quad \dots (2)$$

Case 1 : If $h = k^2$

$$(2) \text{ becomes } u = (c_1 r^k + c_2 r^{-k}) [c_3 \cos(k\theta) + c_4 \sin(k\theta)]$$

Case 2 : If $h = 0$

$$(2) \text{ becomes } u = (c_5 + z c_6) (c_7 + \theta c_8) \\ = [c_5 + (\log r) c_6] [c_7 + \theta c_8]$$

Case 3 :

If $h = -p^2$

$$(2) \text{ becomes } u = (c_9 \cos pz + c_{10} \sin pz) (c_{11} e^{p\theta} + c_{12} e^{-p\theta})$$

Then there are three possible solutions.

$$u = (c_1 r^k + c_2 r^{-k}) [c_3 \cos(k\theta) + c_4 \sin(k\theta)]$$

$$u = (c_5 + c_6 \log r) (c_7 + c_8 \theta)$$

$$u = [c_9 \cos(p \log r) + c_{10} \sin(p \log r)] [c_{11} e^{p\theta} + c_{12} e^{-p\theta}] \quad \text{ans.}$$

Example 16 : The diameter of a semiconductor plate of radius is kept at 0°C and the temperature at the semiconductor boundary is $T^\circ\text{C}$. Find the steady state temperature in the plate.

Solution : Let the centre O of the semicircular plate of the pole and the bounding diameter be as the initial. Let $u(r, \theta)$ be the steady state temperature at any point $p(r, \theta)$ and u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(1)$$

The boundary conditions are

$$(i) u(r, 0) = 0 \quad 0 \leq r \leq a$$

$$(ii) u(r, \pi) = 0 \quad 0 \leq r \leq a$$

$$(iii) u(a, \theta) = T$$

From conditions (ii) and (iii) we have $u \rightarrow 0$ as $r \rightarrow 0$. Hence the appropriate solution of (i) is as solved in example 15.

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \dots(2)$$

Putting $u(r, 0) = 0$ in (2), we get

$$0 = (c_1 r^p + c_2 r^{-p})c_3 \rightarrow c_3 = 0 \quad \dots(3)$$

(2) becomes

$$u = (c_1 r^p + c_2 r^{-p})c_4 \sin p\theta$$

Putting $u(r, \pi) = 0$ in (3), we get

$$0 = (c_1 r^p + c_2 r^{-p})c_4 \sin p\pi \Rightarrow \sin p\pi = 0 = \sin \pi$$

(3) becomes on putting $p = n$

$$u = (c_1 r^n + c_2 r^{-n})c_4 \sin n\theta$$

Since,

$$u = 0 \text{ when } r = 0, \quad 0 = c_2$$

(4) becomes $u = c_1 c_4 r^n \sin n\theta$

The most general solution of (1) is

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \dots(5)$$

Putting $r = a$ and $u = T$ in (5), we have

$$T = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

By Fourier half range series, we get

$$b_n a^n = \frac{2}{\pi} \int_0^{\pi} T \sin n\theta d\theta = \frac{2}{\pi} T \left(\frac{-\cos n\theta}{n} \right)_0^{\pi} = \frac{2T}{n\pi} [-(-1)^n + 1]$$

$$b_n a^n = 0 \quad \text{when } n \text{ is even}$$

$$b_n a^n = \frac{4T}{n\pi} \quad \text{when } n \text{ is odd}$$

$$\Rightarrow b_n = \frac{4T}{n\pi a^n}$$

Hence (5) becomes

$$u(r, \theta) = \frac{4T}{\pi} \left[\frac{r/a}{1} \sin \theta + \frac{(r/a)^3}{3} \sin 3\theta + \frac{(r/a)^5}{5} \sin 5\theta + \dots \right] \text{ Ans.}$$

EXERCISE 3.5 :

- Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at 60°C .
- A semi-circular plate of radius a as its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral of the plate to be insulated.
- Solve the steady-state temperature equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0, \quad 10 \leq r \leq 20, \quad 0 \leq \theta \leq 2\pi \quad \text{subject to}$$

the following conditions $T(10, \theta) = 15 \cos \theta$ and $T(20, \theta) = 30 \sin \theta$

ANSWER 3.5:

$$1. \text{ Ans. } u(r, \theta) = 50 - \frac{200}{n} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{2n-1} \sin(2n-1)\theta$$

$$2. \text{ Ans. } u(r, \theta) = \frac{8k}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{2n-1} \frac{\sin(2n-1)\theta}{(2n-1)^3}$$

$$3. \text{ } T(r, \theta) = \frac{4T}{\pi} \left[\frac{r}{a} \sin \theta + \frac{1}{3} \left(\frac{r}{a}\right)^3 \sin 3\theta + \dots \right]$$

3.8 TRANSMISSION LINE EQUATIONS :

$$\frac{\partial^2 V}{\partial x^2} = RC \frac{\partial V}{\partial t},$$

$$\frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t}$$

are called telegraph equations,

where V = potential i = current, C = capacitance, L = inductance

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}, \quad \frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2}$$

are called radio equations.

Example 17 : Find the current i and voltage v in transmission line of length l , t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $v(x, 0) = v_0 \sin\left(\frac{\pi x}{l}\right)$ and that R and G are negligible.

Solution : $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$

Let $v = XT$ where X and T are the functions of x and t respectively

$$\frac{\partial^2 v}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

$$\text{and } \frac{\partial^2 v}{\partial t^2} = X \frac{d^2 T}{dt^2}$$

$$T \frac{d^2 X}{dx^2} = LCX \frac{d^2 T}{dt^2}, \quad \frac{\frac{d^2 X}{dx^2}}{X} = LC \frac{\frac{d^2 T}{dt^2}}{T} = -p^2 \quad \text{say}$$

Since the initial conditions suggest the values of v and i are periodic functions.

$$X = c_1 \cos px + c_2 \sin px$$

$$T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}$$

$$v = XT$$

$$\text{If } v = (c_1 + \cos px + c_2 \sin px) \left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad \dots(1)$$

when

$$t = 0, v = v_0 \sin \frac{\pi x}{l}$$

...(2)

$$v_0 \sin \frac{\pi x}{l} = (c_1 \cos px + c_2 \sin px) c_3$$

On equating the coefficients, we get

$$c_1 c_3 = 0 \Rightarrow c_1 = 0 \quad \text{and} \quad c_2 c_3 = v_0, p = \frac{\pi}{l}$$

(1) becomes

$$v = \sin \frac{\pi x}{l} \left[v_0 \cos \frac{pt}{\sqrt{LC}} + c_2 c_4 \sin \frac{pt}{\sqrt{LC}} \right]$$

...(3)

Now when $t = 0$, $i = i_0$ (constant)

Hence

$$\frac{\partial i}{\partial x} = 0$$

$$\frac{\partial i}{\partial x} = \frac{-c \partial v}{\partial t}$$

$$\therefore \frac{\partial v}{\partial t} = 0 \text{ when } t = 0$$

Now
$$\frac{\partial v}{\partial t} = \sin \frac{\pi x}{l} \left(\frac{p}{\sqrt{LC}} \right) \left[-v_0 \sin \frac{pt}{\sqrt{LC}} + c_2 c_4 \cos \frac{pt}{\sqrt{LC}} \right] \quad \dots(4)$$

On putting $\frac{\partial v}{\partial t} = 0$ and $t = 0$ in (4) we get $c_2, c_4 = 0 \Rightarrow c_4 = 0$

(3) is reduced to

$$v = v_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{\sqrt{LC}}$$

$$\frac{\partial v}{\partial x} = \frac{\pi}{l} v_0 \cos \frac{\pi x}{l} \cos \frac{\pi t}{\sqrt{LC}} = -L \frac{\partial i}{\partial t} \quad \dots(5)$$

$$\frac{\partial v}{\partial t} = -\frac{v_0 \pi}{l \sqrt{LC}} \sin \frac{\pi x}{l} \sin \frac{\pi t}{\sqrt{LC}} = -\frac{1}{C} \frac{\partial t}{\partial x} \quad \dots(6)$$

Integrating (5) and (6), we get

$$i = -v_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{\sqrt{LC}} + f(x)$$

$$i = -v_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{\sqrt{LC}} + F(t)$$

$\therefore f(x)$ and $F(t)$ must be constant only, since $i = i_0$ when $t = 0$

$\therefore \text{constant} = i_0 = f(x)$

Hence

$$i = i_0 - v_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{\sqrt{LC}}$$

EXERCISE 3.6 :

- Obtain a solution of the telegraph equation $\frac{\partial^2 e}{\partial x^2} = RC \frac{\partial e}{\partial t}$ suitable for the case when e decays with the time and when there is a steady fall of potential from e_0 to 0 along the line of l initially and the sending end is suddenly earthed.

ANSWER 3.6 :

$$1. \quad e(x, t) = \frac{2e_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{CR^2}}$$

Ans.

MULTIPLE CHOICE QUESTION

- If the ends $x = 0$ and $x = L$ are insulated in one dimensional heat flow problems, then the boundary conditions are

(a) $\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(L, t)}{\partial x} = 1, \text{ at } t = 0$ (b) $\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(L, t)}{\partial x} = 0, \text{ at } t = 0$

(c) $\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(L, t)}{\partial x} = 0, \text{ for all } t$ (d) $\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(L, t)}{\partial x} = 1, \text{ for all } t$

- One dimensional heat equation is given by

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(b) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$

- The wave equation is given by

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(b) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial u}{\partial t} + c^2 \frac{\partial^2 u}{\partial x^2}$

- The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is, ..

(a) Two dimensional heat flow

(b) One dimensional heat flow

(c) wave equation

(d) Radio equation

- The equation $\frac{\partial v}{\partial x} = \frac{L}{\partial t} \frac{\partial I}{\partial x}, \frac{\partial I}{\partial x} = C \frac{\partial v}{\partial t}$ is, ..

(a) wave equation

(b) Radio equation

(c) one-dimensional heat flow

(d) two-dimensional heat flow

- The solution of $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ is $u = C_1 r^{\sqrt{h}} + C_2 r^{-\sqrt{h}} [C_3 \sin \sqrt{h} \theta + C_4 \sin \sqrt{h} \theta]$ if $h = k^2$ then solution is

(a) $u = (C_1 r^k + C_2 r^{-k}) [C_3 \cos k \theta + C_4 \sin k \theta]$

(b) $u = (C_1 r^k + C_2 r^{-k}) [C_3 \cos k \theta + C_4 \sin k \theta]$

(c) $u = (C_1 r^k - C_2 r^{-k}) [C_3 \cos k \theta + C_4 \sin k \theta]$

(d) None of these

7. The solution of Laplace equation in polar co-ordinates
 (a) $u = (C_1 r^p + C_2 r^{-p})(C_3 \cos k\theta + C_4 \sin k\theta)$
 (b) $u = [C_5 \cos(p \log r) + C_6 \sin(p \log r)](C_7 e^{r^p} + C_8 e^{-r^p})$
 (c) $u = (C_9 \log r + C_{10})(C_{11} \theta + C_{12})$
 (d) All the above
8. D'Alembert's solution of the wave equation is
 (a) $y = (x, t) = f(x + ct) + f(x - ct)$
 (b) $y(x, t) = f(x + ct) + f(x + ct)$
 (c) $y(x, t) = f(x^2 - ct^2) + f(x - ct)$
 (d) $y(x, t) = f(y - ct) + f(x - ct)$
9. The telegraph equations for potential and current is
 (a) $\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t}$
 (b) $\frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t}$
 (c) both a & b
 (d) None
10. The partial differential equation of the transverse vibrations of string is
 (a) $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$
 (b) $\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 y}{\partial x}$
 (c) $\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 y}{\partial t^2}$
 (d) $\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 x}{\partial y}$
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is
 (a) 0
 (b) 1
 (c) 2
 (d) 3
12. When a vibrating string fastened to two points l_0 apart, has an initial velocity u_0 , its initial conditions is
 (a) $y(0, t) = 0$
 (b) $y(l, t) = 0$
 (c) $\left(\frac{dy}{dt}\right)_{t=0} = 0$
 (d) all of these
14. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$ given $z(x, 0) = 4e^{-3x}$ is,
 (a) $z = 4e^{3x+t}$
 (b) $z = 4e^{-3x+t}$
 (c) $z = 4e^{3x-t}$
 (d) $z = -4e^{3x+t}$
15. The steady state temperature at rod of length l is whose ends are kept at 30° and 40° is
 (a) $u = \frac{10}{l}x + 30$
 (b) $u = \frac{-10}{l}x - 30$

- (c) $u = \frac{10}{l}x - 30$
 (d) None of these
16. The two dimensional steady state heat flow equation in polar co-ordinates is..
 (a) $r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$
 (b) $r \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2}$
 (c) $r^2 \frac{\partial^2 u}{\partial r^2} - r \frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial \theta^2} = 0$
 (d) All these above
17. Solution of $\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$ is..
 (a) $u = 8e^{-12x-3y}$
 (b) $u = 8e^{12x+3y}$
 (c) $u = 8e^{12x}$
 (d) $u = -8e^{-12x-3y}$
18. The general solution of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$ is.....
 (a) $(C_1 \cos px + C_2 \sin px)(C_3 \cos pt + C_4 \sin pt)$
 (b) $(C_1 \cos px - C_2 \sin px)(C_3 \cos pt + C_4 \sin pt)$
 (c) $(C_1 \cos px - C_2 \sin px)(C_3 \cos pt - C_4 \sin pt)$
 (d) None
19. The general solution of the equation $\frac{\partial^2 z}{\partial x \partial y} = 0$ is.....
 (a) $f_1(x) + f_2(y)$
 (b) $f_1(x) + f_2(x)$
 (c) $f_1(y) + f_2(y)$
 (d) $f_1(x) - f_2(y)$
20. The solution of $z(x, y)$ of the equation $\frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0$ is.....
 (a) $f(x + \log y, z) = 0$
 (b) $f(x - \log y, z) = 0$
 (c) $f(x + \log y, x) = 0$
 (d) $f(x + \log y, -z) = 0$

48195

21. The solution of $3x \frac{\partial z}{\partial x} - 5y \frac{\partial z}{\partial y} = 0$ is.....

(a) $f(x^5 y^3, z) = 0$

(c) $f(x^5 y, z) = 0$

(b) $f(x^5 y^3, z^2) = 0$

(d) $f(xy^3, z) = 0$

22. The solution of $\frac{\partial^2 z}{\partial x^2} = \sin(xy)$ is.....

(a) $-\frac{1}{y^2} \sin(xy) + x f_1(y) + f_2(y)$

(c) $-\frac{1}{y^2} \sin(xy) + x f_1(y) - f_2(y)$

(b) $-\frac{1}{y^2} \sin(xy) - x f_1(y) + f_2(y)$

(d) $-\frac{1}{y^2} \sin(xy) - x f_1(y) - f_2(y)$

23. The solution of $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$ if $u(0, t) = u(3, t) = 0$ and $u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi x$ is.....

(a) $(5 \sin 4\pi x e^{-32\pi^2 t} - 3 \sin 8\pi x e^{-128\pi^2 t})$

(c) $(5 \sin 4\pi x e^{-32\pi^2 t} - 3 \sin 8\pi x e^{128\pi^2 t})$

(b) $(5 \sin 4\pi x e^{-32\pi^2 t} + 3 \sin 8\pi x e^{-128\pi^2 t})$

(d) $(5 \sin 4\pi x e^{32\pi^2 t} - 3 \sin 8\pi x e^{-128\pi^2 t})$

24. If the unknown function in a differential equation depends on more than one independent variables then the differential equation is said to be.....

(g) Partial differential equation.

(c) Both (a) and (b)

(b) Ordinary differential equation.

(d) None

ANSWER KEY (MULTIPLE CHOICE QUESTIONS)

01.(c)

02.(b)

03. (a)

04. (a)

05. (b)

06. (c)

07.(d)

08.(a)

09. (c)

10. (a)

11. (a)

12. (d)

13.(d)

14.(a)

15. (a)

16. (a)

17. (a)

18. (a)

19.(a)

20.(a)

21. (a)

22. (a)

23. (a)

24. (a)

4

LAPLACE TRANSFORMS

4.1 INTRODUCTION :

The Laplace transforms is most useful in integral transform in engineering and physical science. Its main contributrus is in the solution of linear ordinary and partial differential equations.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850-1925) to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystemtaic and lacked rigour, which was placed on sound mathematial footing by Bromwich and carson during 1916-1917. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.

4.2 (1) Definition

Let $f(t)$ be a function of t defined for all positive values of t . Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by

(1)

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

(2) Conditions for the existence

The Laplace transform of $f(t)$ i.e. $\int_0^{\infty} e^{-st} f(t) dt$ exists for $s > a$, if

(i) $f(t)$ is continuous

(ii) $Lt \{e^{-at} f(t)\}$ is finite.

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t=0$.

4.3 TRANSFORMS OF BASIC FUNCTIONS

By using definition gives the following formulae :

- (1) $L(1) = \frac{1}{s}$ ($s > 0$)
- (2) $L(e^{at}) = \frac{1}{s-a}$ ($s > a$)
- (3) $L(\sin at) = \frac{a}{s^2 + a^2}$ ($s > 0$)
- (4) $L(\cos at) = \frac{s}{s^2 + a^2}$ ($s > 0$)
- (5) $L(\sinh at) = \frac{a}{s^2 - a^2}$ ($s > |a|$)
- (6) $L(\cosh at) = \frac{s}{s^2 - a^2}$ ($s > |a|$)

Proofs. (1) $L(1) = \int_0^\infty e^{-st} \cdot 1 dt = \left[-\frac{e^{-st}}{s} \right]_0^\infty = \frac{1}{s}$ if $s > 0$

(2) $L(t^n) = \int_0^\infty e^{-st} \cdot t^n dt = \int_0^\infty e^{-p} \cdot \left(\frac{p}{s}\right)^n \frac{dp}{s}$, on putting $st = p$
 $= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} \cdot p^n dp = \frac{\Gamma(n+1)}{s^{n+1}}$, if $n \geq -1$ and $s > 0$

In particular $L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$; $L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

If n be a positive integer, $\Gamma(n+1) = n!$
 therefore, $L(t^n) = n! / s^{n+1}$.

- (3) $L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{s-a}$, if $s > a$
- (4) $L(\sin at) = \int_0^\infty e^{-st} \sin at dt = \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty = \frac{a}{s^2 + a^2}$
- (6) $L(\sinh at) = \int_0^\infty e^{-st} \sinh at dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt$
 $= \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt \right] = \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{a}{s^2 - a^2}$ for $s > |a|$

4.4 BASIC PROPERTIES OF LAPLACE TRANSFORMS

Theorem 1. Linearity property. If p, q, r be any constants and f, g, h any functions of t , then

$$L[pf(t) + qg(t) - rh(t)] = pL\{f(t)\} + qL\{g(t)\} - rL\{h(t)\}$$

for by definition

$$\begin{aligned} L.H.S. &= \int_0^\infty e^{-st} \{pf(t) + qg(t) - rh(t)\} dt \\ &= p \int_0^\infty e^{-st} f(t) dt + q \int_0^\infty e^{-st} g(t) dt - r \int_0^\infty e^{-st} h(t) dt = pL\{f(t)\} + qL\{g(t)\} - rL\{h(t)\} \end{aligned}$$

This result can easily be generalised

Because of the above property of L , it is called a linear operator.

Theorem 2. First shifting property. If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at} f(t)\} = \bar{f}(s-a)$$

By definition, $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt$
 $= \int_0^\infty e^{-rt} f(t) dt$, where $r = s-a = \bar{f}(r) = \bar{f}(s-a)$

This, if we know the transform $\bar{f}(s)$ of $f(t)$, we can write the transform of $e^{at} f(t)$

simply replacing s by $s-a$ to get $\bar{f}(s-a)$.

$$(1) \quad L(e^{at}) = \frac{1}{s-a}$$

$$\left[\because L(1) = \frac{1}{s} \right]$$

$$(2) \quad L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$\left[\because L(t^n) = \frac{n!}{s^{n+1}} \right]$$

$$(3) \quad L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$\left[\because L(\sin bt) = \frac{b}{s^2 + b^2} \right]$$

$$(4) \quad L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

$$\left[\because L(\cos bt) = \frac{s}{s^2 + b^2} \right]$$

$$(5) \quad L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$$\left[\because L(\sinh bt) = \frac{b}{s^2 - b^2} \right]$$

$$(6) \quad L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

$$\left[\because L(\cosh bt) = \frac{s}{s^2 - b^2} \right]$$

where in each case $s > a$.

Example 1 Find the Laplace transforms of

- (i) $\cos t \cos 2t$ (ii) $\cos^2 2t$ (iii) $\sin^3 2t$

Solu. (i) Since $\cos t \cos 2t = \frac{1}{2} [\cos 3t + \cos t]$

$$\therefore L(\cos t \cos 2t) = \frac{1}{2} [L(\cos 3t) + L(\cos t)] = \frac{1}{2} \left[\frac{s}{s^2 + 3^2} + \frac{s}{s^2 + 1^2} \right] = \frac{s(s^2 + 5)}{(s^2 + 9)(s^2 + 1)}$$

$$(ii) \text{ Since } \cos^2 2t = \frac{1}{2} (1 + \cos 4t)$$

$$\therefore L(\cos^2 2t) = \frac{1}{2} \{L(1) + L(\cos 4t)\} = \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 16} \right\}$$

(iii) Since

$$\sin 6t = 3 \sin 2t - 4 \sin^3 2t$$

$$\text{or } \sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$$

$$\therefore L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{6}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}$$

Example 2 Find the Laplace transforms of

- (i) $e^{-3t} (2 \cos 5t - 3 \sin 5t)$ (ii) $e^{2t} \cos^2 t$ (iii) $e^{4t} \sin 2t \cos 2t$

Solu. (i) $L(e^{-3t} (2 \cos 5t - 3 \sin 5t)) = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s-9}{s^2 + 6s + 34}$$

$$(ii) \text{ Since } L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 4} \right\}$$

$\therefore$ By shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2} \left\{ \frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4} \right\}$$

$$(iii) \text{ Since } L(\sin 2t \cos t) = \frac{1}{2} L(\sin 3t + \sin t) = \frac{1}{2} \left\{ \frac{3}{s^2 + 3^2} + \frac{1}{s^2 + 1^2} \right\}$$

$\therefore$ By shifting property, we obtain

$$L(e^{4t} \sin 2t \cos t) = \frac{1}{2} \left\{ \frac{3}{(s-4)^2 + 9} + \frac{1}{(s-4)^2 + 1} \right\}$$

Example 3 If $L f(t) = \bar{f}(s)$, show that

$$L[\sinh at] f(t) = \frac{1}{2} [\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[\cosh at f(t)] = \frac{1}{2} [\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (i) $\sinh 2t \sin 3t$ (ii) $\cosh 3t \cos 2t$

Soln. We have $L[\sinh at f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at})f(t)\right] = \frac{1}{2}[L\{e^{at}f(t)\} - L\{e^{-at}f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)], \text{ by shifting property}$$

Similarly

$$L[(\cosh at)f(t)] = \frac{1}{2}[L\{e^{at}f(t)\} + L\{e^{-at}f(t)\}]$$

$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)], \text{ by shifting property.}$$

(i) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2} \left\{ \frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2} \right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(ii) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2} \left\{ \frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2} \right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}$$

Example.4 Show that

$$(i) L(t \sin at) = \frac{2as}{(s^2 + a^2)^2}$$

$$(ii) L(te^{iat}) = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$$

or $L[t(\cos at + i \sin at)] = \frac{(s^2 - a^2) + i(2as)}{(s^2 + a^2)^2}$

Equating the real and imaginary parts from both sides, we get the desired results.

Example.5 Find the Laplace transform of $f(t)$ defined as

$$f(t) = t/\tau, \text{ when } 0 < t < \tau$$

$$= 1, \text{ when } t > \tau$$

Soln. $Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} \cdot 1 dt = \frac{1}{\tau} \left[t \cdot \frac{e^{-st}}{-s} \Big|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s} dt \right] + \left[\frac{e^{-st}}{-s} \right]_\tau^\infty$

$$= \frac{1}{\tau} \left[\frac{\tau e^{-s\tau} - 0}{-s} - \left[\frac{e^{-st}}{-s^2} \right]_0^\tau \right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau} - 1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1 - e^{-s\tau}}{\tau s^2}$$

Example.6 Find the Laplace transform of $F(t) = \begin{cases} 1, & 0 < t \leq 1 \\ t, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}$

Soln. $L\{F(t)\} = \int_0^1 e^{-st} 1 dt + \int_1^2 e^{-st} t dt + \int_2^\infty e^{-st} (0) dt$

$$= \left[\frac{e^{-st}}{-s} \right]_0^1 + \left[t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^2$$

$$= \frac{1 - e^{-s}}{s} + \left\{ \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} \right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} \right) \right\}$$

$$= \frac{1}{2} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

Example.7 Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$

Soln. Since $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$$\therefore L\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$

$$= \frac{\Gamma(3/2 + 1)}{s^{3/2 + 1}} - 3 \frac{\Gamma(1/2 + 1)}{s^{1/2 + 1}} + 3 \frac{\Gamma(-1/2 + 1)}{s^{-1/2 + 1}} - \frac{\Gamma(-3/2 + 1)}{s^{-3/2 + 1}}$$

$$\begin{aligned}
 &= \frac{3}{2} \Gamma\left(\frac{3}{2}\right) - 3 \frac{1}{2} \Gamma\left(\frac{1}{2}\right) + 3 \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} \\
 &= \frac{3\sqrt{\pi}}{4s^{5/2}} - \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + 2\sqrt{\pi}\sqrt{s} \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \right] \\
 &= \frac{\sqrt{\pi}}{4} \left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}} \right)
 \end{aligned}$$

Theorem 4. Changes of scale property. If $L\{f(t) = \bar{f}(s)\}$, then $L\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

$$\begin{aligned}
 L\{f(at)\} &= \int_0^\infty e^{-st} f(at) dt = \int_0^\infty e^{-su/a} f(u) \cdot du/a \quad \left| \begin{array}{l} \text{Put } at = u \\ dt = du/a \end{array} \right. \\
 &= \frac{1}{a} \int_0^\infty e^{-su/a} f(u) du = \frac{1}{a} \bar{f}(s/a)
 \end{aligned}$$

Example .8 Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$

Soln. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a} \tan^{-1}\left\{\frac{1}{s/a}\right\} = \frac{1}{a} \tan^{-1}\left(\frac{a}{s}\right) \text{ i.e. } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$$

Find the Laplace transform of

1. $1 + 2\sqrt{t} + 3/\sqrt{t}$

4. $(\sin t - \cos t)^2$

7. $t^2 e^{-2t}$

EXERCISE 4.1 :

2. $3 \cosh at - \cos at$

5. $\sin 2t \cos 3t$

8. $f(x) = \begin{cases} 1, & 0 \leq x \leq a/2 \\ -1, & a/2 < x \leq a \end{cases}$

3. $\cos(at+b)$

6. $e^{-at} \sinh bt$

9. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

ANSWERS 4.1 :

1. $\frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \sqrt{\frac{\pi}{s}}$

2. $\frac{2a^2 s}{s^4 - a^4}$

3. $\frac{s \cos b - a \sin b}{s^2 + a^2}$

4. $\frac{s^2 - 2s + 4}{s(s^2 + 4)}$

5. $\frac{2(s^2 - 5)}{(s^2 + 1)(s^2 + 25)}$

6. $\frac{b}{(s+a)^2 - b^2}$

7. $\frac{2}{(s+2)^3}$

8. $\frac{1}{s} \left(1 + e^{-as} - 2e^{-\frac{as}{2}} \right)$

9. $\frac{1 + e^{-\pi s}}{s^2 + 1}$

LAPLACE TRANSFORMS OF FOLLOWING FUNCTION

4.5 PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T , i.e. $f(t+T) = f(t)$, then

$$Lf(t) = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^T e^{-st} f(t) dt + \int_T^{2T} e^{-st} f(t) dt + \int_{2T}^{3T} e^{-st} f(t) dt + \dots$$

In the second integral put $t = u + T$, in the third integral put $u = u + 2T$, and so on. Then

$$\begin{aligned}
 L\{f(t)\} &= \int_0^T e^{-st} f(t) dt = \int_0^T e^{-s(u+T)} f(u+T) du + \int_0^T e^{-s(u+2T)} f(u+2T) du + \dots \\
 &= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-su} f(u) du + e^{-2sT} \int_0^T e^{-su} f(u) du + \dots \\
 &\quad [\because f(u) = f(u+T) = f(u+2T) \text{ etc}]
 \end{aligned}$$

$$= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-st} f(t) dt + e^{-2sT} \int_0^T e^{-st} f(t) dt + \dots$$

$$= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

Example 9 Find the Laplace transforms of the function

$$f(t) = \sin \omega t, \quad 0 < t < \pi / \omega$$

$$= 0, \quad \pi / \omega < t < 2\pi / \omega$$

Soln. Since $f(t)$ is a periodic function with period $2\pi / \omega$.

$$\therefore L\{f(t)\} = \frac{1}{1 - e^{-2\pi s / \omega}} \int_0^{2\pi / \omega} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2\pi s / \omega}} \left[\int_0^{\pi / \omega} e^{-st} \sin \omega t dt + \int_{\pi / \omega}^{2\pi / \omega} e^{-st} \cdot 0 dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s / \omega}} \left[\frac{e^{-st} (-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right]_0^{\pi / \omega}$$

$$= \frac{\omega e^{-\pi s / \omega} + \omega}{(1 - e^{-2\pi s / \omega})(s^2 + \omega^2)} = \frac{\omega}{(1 - e^{-\pi s / \omega})(s^2 + \omega^2)}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{\pi}{s} (e^{-2\pi s} - e^{\pi s}) + \frac{1}{s^2} (1 + e^{-2\pi s} - 2e^{-\pi s}) \right\}$$

4.6 SPECIAL FUNCTIONS

(1) **Transforms of Bessel function $J_0(x)$ and $J_1(x)$**

We know that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$\therefore L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \frac{6!}{s^7} + \dots$$

$$= \frac{1}{s} \left\{ 1 - \frac{1}{2} \left(\frac{1}{s^2} \right) + \frac{1.3}{2.4} \left(\frac{1}{s^4} \right) - \frac{1.3.5}{2.4.6} \left(\frac{1}{s^6} \right) + \dots \right\}$$

$$= \frac{1}{5} \left(1 + \frac{1}{s^2} \right)^{-1/2} = \frac{1}{\sqrt{(s^2 - 1)}} \quad (1)$$

Also since $J_0'(x) = -J_1(x)$

$$\therefore L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{(s^2 - 1)}} \quad (2)$$

(2) Transform of Error Function

We know that $\operatorname{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots \right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \dots \right)$$

$$\therefore L\{\operatorname{erf}(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{ \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!s^{7/2}} - \frac{\Gamma(9/2)}{7.3!s^{9/2}} + \dots \right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2} \frac{1}{s^{5/2}} + \frac{1.3}{2.4} \frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6} \frac{1}{s^{9/2}} + \dots$$

$$= \frac{1}{s^{3/2}} \left\{ 1 - \frac{1}{2} \frac{1}{s^2} + \frac{1.3}{2.4} \frac{1}{s^2} - \frac{1.3.5}{2.4.6} \frac{1}{s^2} + \dots \right\}$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s} \right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}} \quad (3)$$

Example 10 Evaluate (i) $\{e^{-at} J_0(at)\}$ (ii) $L\{\operatorname{erf} 2\sqrt{t}\}$

Soln. (i) We know that $L\{J_0(at)\} = \frac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{(s+a^2) + a^2}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(ii) We know that $L(\operatorname{erf} \sqrt{t}) = \frac{1}{s(s+1)}$

$$\therefore L(\operatorname{erf} 2\sqrt{t}) = L(\operatorname{erf} \sqrt{4t}) = \frac{1}{4} \cdot \frac{1}{\sqrt{\frac{s}{4} + 1}} = \frac{2}{s\sqrt{s+4}}$$

EXERCISE 4.2 :

- Find the Laplace transform of the saw-toothed wave of period T , given $f(t) = t/T$ for $0 < t < T$
- Find the Laplace transform of the triangular wave of period $2a$ given by $f(t) = t, 0 < t < a$
 $= 2a - t, a < t < 2a$
- Find the Laplace transform of the following functions :
 - $J_0(ax)$
 - $e^{2t} \operatorname{erf}(\sqrt{t})$

ANSWERS 4.2 :

- $(1/s^2T) - e^{-sT}/s(1 - e^{-sT})$
- $(1/s^2) \tanh \frac{1}{2}as$
- (i) $\frac{1}{\sqrt{s^2 + a^2}}$ (ii) $\frac{1}{(s-2)\sqrt{s+1}}$

4.7 DERIVATIVES

- (1) If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.
- $$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt.$$

$$= \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t) dt$$

Now assuming $f(t)$ to be such that $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be exponential order s .

Thus, $L\{f'(t)\} = f(0) + s \int_0^\infty e^{-st} f(t) dt$

Whence follows the desired result.

(2) If $f'(t)$ and its first $(n-1)$ derivatives be condition, then

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

Using the general rule of integration by parts.

$$\begin{aligned} L\{f^n(t)\} &= \int_0^\infty e^{-st} f^n(t) dt \\ &= \left[e^{-st} f^{n-1}(t) - (-s) e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \dots + (-1)^{n-1} (-s)^{n-1} e^{-st} f(t) \right]_0^\infty \\ &\quad + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t) dt \\ &= -f^{n-1}(0) - s f^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t) dt \end{aligned}$$

Assuming that $\lim_{t \rightarrow \infty} e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \dots, n-1$

This proves the required result.

4.8.

INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u) du\right\} = \frac{1}{s} \bar{f}(s)$

Let

$$\phi(t) = \int_0^t f(u) du, \text{ then } \phi'(t) = f(t) \text{ and } \phi(0) = 0$$

$\therefore$

$$L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0)$$

or

$$\bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\} \text{ i.e. } L\left\{\int_0^t f(u) du\right\} = \frac{1}{s} \bar{f}(s)$$

4.9

MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [\bar{f}(s)], \text{ where } n = 1, 2, 3, \dots$$

$$\text{We have } \int_0^\infty e^{-st} f(t) dt = \bar{f}(s)$$

Differentiating both sides with respect to s , $\frac{d}{ds} \left\{ \int_0^\infty e^{-st} f(t) dt \right\} = \frac{d}{ds} \{\bar{f}(s)\}$
or by Leibnitz's rule for differentiation under the integral sign.

$$\int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t) dt = \frac{d}{ds} \{\bar{f}(s)\}$$

$$\text{or } \int_0^\infty \left\{ -te^{-st} f(t) \right\} dt = \frac{d}{ds} \{\bar{f}(s)\} \quad \text{or} \quad \int_0^\infty -e^{-st} [tf(t)] dt = -\frac{d}{ds} \{\bar{f}(s)\}$$

which proves the theorem for $n=1$

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st} [t^m f(t)] dt = (-1)^m \frac{d^m}{ds^m} [\bar{f}(s)]$$

$$\text{Then } \frac{d}{ds} \left[\int_0^\infty e^{-st} t^m f(t) dt \right] = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)]$$

$$\text{or by Leibnitz's rule } \int_0^\infty (-te^{-st}) t^m f(t) dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)]$$

$$\text{or } \int_0^\infty e^{-st} [t^{m+1} f(t)] dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)]$$

This shows that, if the theorem is true for $n = m$, it is also true for $n = m+1$. But it is true for $n = 1$. Hence it is true for $n = 1+1 = 2$, and $n = 2+1 = 3$ and so on. Thus the theorem is true for all positive integral values of n .

Example 11. Find the Laplace transforms of

(i) $t \cos at$

(ii) $t^2 \sin 2t$

(iii) $t^3 e^{-3t}$

(iv) $te^{-t} \sin 3t$

Soln. (i) Since

$$L(\cos at) = s / (s^2 + a^2)$$

$$\therefore L(t \cos at) = -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) = -\frac{s^2 + a^2 - s \cdot 2s}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

(ii) Since

$$\sin 2t = \frac{a}{s^2 + a^2}$$

$$\therefore L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2} \left(\frac{2}{s^2 + 2^2} \right) = \frac{d}{ds} \left\{ \frac{-4s}{(s^2 + 2^2)^2} \right\} = \frac{4(3s^2 - 2^2)}{(s^2 + 2^2)^3}$$

(iii) Since

$$L(e^{-3t}) = 1/(s+3),$$

$$\therefore L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3} \left(\frac{1}{s+3} \right) = -\frac{(-1)^3 \cdot 3!}{(s+3)^{3+1}} = 6/(s+3)^4$$

$$(iv) \text{ Since } L(\sin 3t) = \frac{3}{s^2 + 3^2}, \text{ therefore } L(t \sin 3t) = -\frac{d}{ds} \left(\frac{3}{s^2 + 3^2} \right) = \frac{6s}{(s^2 + 9)^2}$$

Now using the shift property.

$$L(e^{-t} t \sin 3t) = \frac{6(s+1)}{[(s+1)^2 + 9]^2} = \frac{6(s+1)}{(s^2 + 2s + 10)^2}$$

Example 12 Evaluate (i) $L\{tJ_0(at)\}$

Soln.

$$(i) \text{ Since } L\{J_0(at)\} = \frac{1}{\sqrt{s^2 + a^2}}$$

$$\therefore L\{tJ_0(at)\} = -\frac{d}{ds} \left\{ L\{J_0(at)\} \right\} = -\frac{d}{ds} \left(\frac{1}{\sqrt{s^2 + a^2}} \right) = \frac{s}{(s^2 + a^2)^{3/2}}$$

(ii) $L\{tJ_1(t)\}$ (iii) $L\{t \operatorname{erf} 2\sqrt{t}\}$

$$(ii) \text{ Since } L\{J_1(t)\} = 1 - \frac{s}{\sqrt{(s^2+1)}}$$

$$\therefore L\{tJ_1(t)\} = -\frac{d}{ds}\{LJ_1(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{\sqrt{(s^2+1)}}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

$$(iii) \text{ Since, } L(\operatorname{erf} \sqrt{t}) = \frac{1}{s\sqrt{(s+1)}}$$

$$\therefore L(\operatorname{erf} 2\sqrt{t}) = L \operatorname{erf} \sqrt{4t} = \frac{1}{4} \cdot \frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

$$\text{Thus } L(t \operatorname{erf} 2\sqrt{t}) = -\frac{d}{ds}\left\{\frac{2}{s\sqrt{(s+4)}}\right\} = -\frac{d}{ds}\left\{\frac{2}{\sqrt{(s^3+4s^2)}}\right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

4.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{1}{t}f(t)\right\} = \int_0^\infty \bar{f}(s) ds$ provided the integral exists.

We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$

Integrating both sides with respect to s from s to ∞ .

$$\int_s^\infty \bar{f}(s) ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t) dt \right] ds$$

$$= \int_0^\infty \int_s^\infty f(t) e^{-st} ds dt$$

$$= \int_0^\infty f(t) \left(\int_s^\infty e^{-st} ds \right) dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t} \right]_s^\infty dt = \int_0^\infty e^{-st} \cdot \frac{f(t)}{t} dt = L\left\{\frac{1}{t}f(t)\right\}$$

[Changing the order of integration]

[$\because t$ is independent of s]

Example 13 Find the Laplace transform of (i) $(1-e^t)/t$ (ii) $\frac{\cos at - \cos bt}{t}$

$$\text{Soln. (i) Since } L(1-e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \left[\log s - \log(s-1) \right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right) \right]_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

$$(ii) \text{ Since } L(\cos at - \cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\therefore L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds = \left[\frac{1}{2} \log(s^2+a^2) - \frac{1}{2} \log(s^2+b^2) \right]_s^\infty$$

$$= \frac{1}{2} \lim_{s \rightarrow \infty} \log \frac{s^2+a^2}{s^2+b^2} - \frac{1}{2} \log \frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2} \log\left(\frac{1+0}{1+0}\right) - \frac{1}{2} \log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} \quad [\because \log 1 = 0]$$

Example 14. Evaluate (i) $\left\{e^{-t} \int_0^t \frac{\sin t}{t} dt\right\}$ (ii) $L\left\{t \int_0^t \frac{e^{-t} \sin t}{t} dt\right\}$ (iii) $L\left\{\int_0^t \int_0^t \int_0^t (t \sin t) dt dt dt\right\}$

$$\text{Soln. (i) We have that } L(\sin t) = \frac{1}{s^2+1}$$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1} ds = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore L\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s} \cot^{-1} s$$

Thus by shifting property, $L\left\{e^{-t} \left(\int_0^t \frac{\sin t}{t} dt\right)\right\} = \frac{1}{s+1} \cot^{-1}(s+1)$

(ii) Since $L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$

$\therefore L\left(e^{-t} \frac{\sin t}{t}\right) = \cot^{-1}(s+1)$

and $L\left\{\int_0^t e^{-t} \frac{\sin t}{t} dt\right\} = \frac{1}{s} \cot^{-1}(s+1)$

Hence $L\left\{t \int_0^t e^{-t} \frac{\sin t}{t} dt\right\} = -\frac{d}{ds} \left\{ \frac{\cot^{-1}(s+1)}{s} \right\}$
 $= \frac{s \cdot \left[\frac{-1}{1+(s+1)^2} \right] - \cot^{-1}(s+1)}{s^2} = \frac{s + (s^2 + 2s + 2) \cot^{-1}(s+1)}{s^2 (s^2 + 2s + 2)}$

(iii) Since $L(\sin t) = \frac{1}{s^2 + 1}$

$\therefore L(t \sin t) = -\frac{d}{ds} \frac{1}{(s^2 + 1)} = \frac{2s}{(s^2 + 1)^2}$

Thus $L\left\{\int_0^t \int_0^t \int_0^t (t \sin t) dt \cdot dt \cdot dt\right\} = \frac{1}{s^3} L(t \sin t) = \frac{1}{s^3} \cdot \frac{2s}{(s^2 + 1)^2} = \frac{2}{s^2 (s^2 + 1)^2}$

4.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example.15 Evaluate (i) $\int_0^\infty t e^{-3t} \sin t dt$ (ii) $\int_0^\infty \frac{\sin mt}{t} dt$ (iii) $L\left\{\int_0^\infty \frac{e^t \sin t}{t} dt\right\}$

Soln. (i) $\int_0^\infty t e^{-3t} \sin t dt = \int_0^\infty e^{-st} (t \sin t) dt$ where $s=3$

$= L(t \sin t)$, by definition

$= (-1) \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2} = \frac{2 \times 3}{(3^2 + 1)^2} = \frac{3}{50}$

(ii) Since $L(\sin mt) = m / (s^2 + m^2) = f(s)$, say

$L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s) ds = \int_s^\infty \frac{m ds}{s^2 + m^2} = \left[\tan^{-1} \frac{s}{m} \right]_s^\infty$

or by Def. $\int_0^\infty e^{-st} \frac{\sin mt}{t} dt = \frac{\pi}{2} - \tan^{-1} \frac{s}{m}$

Now $\lim_{s \rightarrow 0} \tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$

$\lim_{s \rightarrow 0} \tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$

$\int_0^\infty \frac{\sin mt}{t} dt = \frac{\pi}{2}$ if $m > 0$ or $\pi/2$ if $m < 0$

(iii) Since $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

$\therefore L\left\{e^t \left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1)$, by shifting property.

Thus $L\left[\int_0^\infty \left\{e^t \left(\frac{\sin t}{t}\right)\right\} dt\right] = \frac{1}{s} \cot^{-1}(s-1)$

EXERCISE 4.3 :

1. Find $L\left(\int_0^t e^{-t} \cos t dt\right)$

2. $f \sin^2 t$

3. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$

4. $\sin 2t - 2t \cos 2t$

5. $t^2 \cos at$

6. $t \sinh at$

7. $(e^{-at} - e^{-bt})/t$

8. Prove that (i) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a}$

(ii) $\int_0^\infty \frac{e^{-t} \sin^2 t}{t} dt = \frac{1}{4} \log 5$

ANSWERS.4.3

1. $\frac{s+1}{s(s^2+2s+2)}$

2. $\frac{2(3s^2+4)}{s^2(s^2+4)^2}$

4. $\frac{16}{(s^2+4)^2}$

5. $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$

6. $\frac{2as}{(s^2-a^2)}$

7. $\log \frac{(s+b)}{(s+a)}$

4.12 INVERSE TRANSFORMS BY PRATIAL FRACTIONS

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)!}$

4. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, \dots$

5. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a} \sin at$

7. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a} \sinh at$

8. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

9. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cos bt$

10. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \sin bt$

11. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at$

12. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2} (\sin at - at \cos at)$

above results is very simple proof are omitted.

Example.16 Find the inverse transforms of

(i) $\frac{s^2-3s+4}{s^3}$

(ii) $\frac{s+2}{s^2-4s+13}$

Soln. (i) $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right) = L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right) = 1 - 3t + 4 \cdot t^2 / 2! = 1 - 3t + 2t^2$

$$\begin{aligned} \text{(ii)} \quad L^{-1}\left(\frac{s+2}{s^2-4s+3}\right) &= L^{-1}\left(\frac{s+2}{(s-2)^2+9}\right) = L^{-1}\left(\frac{s-2+4}{(s-2)^2+3^2}\right) \\ &= L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2+3^2}\right) = e^{2t} \cos 3t + \frac{4}{3} e^{2t} \sin 3t \end{aligned}$$

Example.17 Find the inverse transforms of

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

(ii) $\frac{4s+5}{(s-1)^2(s+2)}$

Soln. (i) Here the denominator $= (s-1)(s-2)(s-3)$

Sol. let $\frac{2s^2-6s+6}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

Then $A = [2 \cdot 1^2 - 6 \cdot 1 + 5] / (1-2)(1-3) = \frac{1}{2}$

$B = [2 \cdot 2^2 - 6 \cdot 2 + 5] / (2-1)(2-3) = -1$

and $C = [2 \cdot 3^2 - 6 \cdot 3 + 5] / (3-1)(3-2) = \frac{5}{2}$

$$\begin{aligned} \therefore L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) &= \frac{1}{2} L^{-1}\left(\frac{1}{s-1}\right) = L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s-3}\right) \\ &= \frac{1}{2} e^t - e^{2t} + \frac{5}{2} e^{3t} \end{aligned}$$

$$\text{(ii) Let } \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, $4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$

Putting $s=1$, $9=3B$, $\therefore B=3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \frac{1}{3}, \quad \therefore A = \frac{1}{3}$$

$$\therefore L^{-1} \left[\frac{4s+5}{(s-1)^2(s+2)} \right] = \frac{1}{3} L^{-1} \left(\frac{1}{s-1} \right) + 3L^{-1} \left[\frac{1}{(s-1)^2} \right] = \frac{1}{3} L^{-1} \left(\frac{1}{s+2} \right)$$

$$= \frac{1}{3} e^t + 3te^t - \frac{1}{3} e^{-2t}$$

Example.18 Find the inverse transforms of

$$(i) \frac{5s+3}{(s-1)(s^2+2s+5)}$$

$$(ii) \frac{s}{s^4+4a^4}$$

Soln. (i) Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{5(1)+3}{(s-1)(1^2+2.1+5)} + \frac{As+B}{s^2+2s+5}$

Multiplying both sides by $(s-1)(s^2+2s+5)$

$$5s+3 = 1 \cdot (s^2+2s+5) + (As+B)(s-1)$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A,$$

$$\therefore A = -1$$

Putting $s=0, 3=5-B,$

$$\therefore B = 2$$

$$\therefore L^{-1} \left[\frac{5s+3}{(s-1)(s^2+2s+5)} \right] = L^{-1} \left(\frac{1}{s-1} \right) + L^{-1} \left(\frac{-s+2}{s^2+2s+5} \right)$$

$$= L^{-1} \left(\frac{1}{s-1} \right) + L^{-1} \left[\frac{-(s+1)+3}{(s+1)^2+4} \right]$$

$$= L^{-1} \left(\frac{1}{s-1} \right) - L^{-1} \left[\frac{s+1}{(s+1)^2+2^2} \right] + 3L^{-1} \left[\frac{1}{(s+1)^2+2^2} \right]$$

$$= e^t - e^{-t} \cos 2t + \frac{3}{2} e^{-t} \sin 2t$$

$$(ii) \text{ Since } s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$$

$$\therefore \text{Let } \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

Multiplying both sides by $s^4+4a^4,$

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

(i)

Equating coefficients $s^3, 0 = A + C$

(ii)

Equating coefficients of $s^2, 0 = -2aA + B + 2aC + D$

(iii)

Equating coefficients of $s, 1 = 2a^2A - 2aB + 2a^2C + 2aD$

(iv)

Putting $s=0,$

$$0 = 2a^2B + 2a^2D$$

(v)

From (iv),

$$B + D = 0$$

$\therefore$ (ii) becomes $-A + C = 0$, and by (i), we get $A = C = 0$

Then (iii) reduces to $D - B = 1/2a$ and by (v), $B = -1/4a, D = 1/4a$

$$\therefore L^{-1} \left(\frac{s}{s^4+4a^4} \right) = \frac{1}{4a} L^{-1} \left(\frac{1}{s^2+2as+2a^2} \right) + \frac{1}{4a} L^{-1} \left(\frac{1}{s^2-2as+2a^2} \right)$$

$$= -\frac{1}{4a} L^{-1} \left[\frac{1}{(s+a)^2+a^2} \right] + \frac{1}{4a} L^{-1} \left[\frac{1}{(s-a)^2+a^2} \right]$$

$$= -\frac{1}{4a} \cdot \frac{1}{a} e^{-at} \sin at + \frac{1}{4a} \cdot \frac{1}{a} e^{at} \sin at = \frac{1}{2a^2} \sin at \left(\frac{e^{at} - e^{-at}}{2} \right) = \frac{1}{2a^2} \sin at \sinh at$$

EXERCISE 4.4:

Find the inverse Laplace transforms of:

1. $\frac{3(s^2-2)^2}{2s^5}$

2. $\frac{3s}{s^2+2s-8}$

3. $\frac{3s+2}{s^2-s-2}$

$$4. \frac{s^2 + s - 2}{s(s+3)(s-2)}$$

$$5. \frac{2p^2 - 6p + 5}{p^3 - 6p^2 + 11p - 6}$$

$$6. \frac{s}{(s^2 - 1)^2}$$

$$7. \frac{1+2s}{(s+2)^2(s-1)^2}$$

$$8. \frac{s}{(s+1)^2(s^2+1)}$$

$$9. \frac{s^3}{s^4 - a^4}$$

ANSWERS 4.4

$$1. \frac{3}{2} - 3t^2 + \frac{1}{4}t^4$$

$$2. e^{2t} + 2e^{-t}$$

$$3. \frac{8e^{2t}}{3} + \frac{e^{-t}}{3}$$

$$4. \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

$$5. \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

$$6. \frac{1}{2} \tanh t$$

$$7. \frac{1}{3}t(e^t - e^{-2t})$$

$$8. \frac{1}{2}(\sin t - te^{-t})$$

$$9. \frac{1}{2}(\cos at + \cosh at)$$

4.13. METHODS OF FINDING INVERSE TRANSFORM

The most effective method of finding the inverse transforms which depend on the following important inversion formulae.

I. Shifting property for inverse Laplace transforms.

If $L^{-1}[\bar{f}(s)] = f(t)$, then

$$L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)]$$

II. If $L^{-1}[\bar{f}(s)] = f(t)$ and $f(0) = 0$, then

$$L^{-1}\{s\bar{f}(s)\} = \frac{d}{dt}\{f(t)\}$$

In general, $L^{-1}\{s^n \bar{f}(s)\} = \frac{d^n}{dt^n}\{f(t)\}$ provided $f(0) = f'(0) = \dots = f^{n-1}(0) = 0$

III. If $L^{-1}[\bar{f}(s)] = f(t)$, then

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t) dt$$

Also

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{ \int_0^t f(t) dt \right\} dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{ \int_0^t \left\{ \int_0^t f(t) dt \right\} dt \right\} dt \text{ as so on.}$$

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then

$$t f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L[t f(t)] = -\frac{d}{ds}\{\bar{f}(s)\}$

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty \bar{f}(s) ds$$

Is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transforms of $\int_0^\infty \bar{f}(s) ds$ can be conveniently calculated.

Example 19 Find the inverse Laplace transforms of the following.

$$(i) \frac{s^2}{(s^2 - 2)^3} \quad (ii) \frac{s+3}{s^2 - 4s + 13} \quad (iii) \frac{(s+2)^2}{(s^2 + 4s + 8)^2}$$

Soln.

$$(i) \text{ Since } s^2 = (s-2)^2 + 4(s-2) + 4$$

$$\therefore \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}t + 2e^{2t}t^2.$$

[Using shifting property]

$$(ii) \frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3} L^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$

$$= e^{2t} \cos 3t + \frac{5}{3} e^{2t} \sin 3t$$

[Using shifting property]

$$(iii) L^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+8)}\right\} = L^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+4+4)}\right\} = L^{-1}\left\{\frac{(s+2)^2}{[(s+2)^2+4]}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t} L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\} = \frac{e^{-2t} \sin 2t}{2} - 4e^{-2t} L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t} \sin 2t}{2} - 4e^{-2t} \left\{\frac{1}{4} \left(\frac{\sin 2t}{4} - \frac{t \cos 2t}{2}\right)\right\}$$

$$= e^{-2t} \left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t \cos 2t}{2}\right\} = e^{-2t} \left\{\frac{\sin 2t}{4} + \frac{t \cos 2t}{2}\right\}$$

Example 20 Find the inverse transform of (i) $1/s(s^2+a^2)$ (ii) $1/s(s+a)^3$

Soln. (i) Since $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \sin at$

Therefore by formula III above,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a} \sin at \, dt = \frac{1}{a^2} [-\cos at]_0^t = (1 - \cos at) / a^2$$

$$(ii) L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\} = e^{-at} L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$$

$$\text{Now } L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \therefore L^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{at} \, dt = \frac{e^{at}}{a} - \frac{1}{a}, \text{ by III above}$$

$$\therefore L^{-1}\left\{\frac{1}{(s-a)s^2}\right\} = \frac{1}{a} \int_0^t (e^{at} - 1) \, dt = \frac{1}{a^2} (e^{at} - at - 1)$$

$$L^{-1}\left\{\frac{1}{(s-a)s^3}\right\} = \frac{1}{a^2} \int_0^t (e^{at} - at - 1) \, dt = \frac{1}{a^3} \left(e^{at} - \frac{a^2}{2} t^2 - at - 1\right)$$

$$\text{Hence } L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = e^{-at} \cdot \frac{1}{a^3} \left(e^{at} - \frac{a^2}{2} t^2 - at - 1\right) = \frac{1}{a^3} \left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2} t^2 e^{-at}\right)$$

Example 21 Find the inverse Laplace transform of

$$(i) \frac{s}{(s^2+a^2)^2} \quad (ii) \frac{s^2}{(s^2+a^2)^2} \quad (iii) \frac{1}{(s^2+a^2)^2}$$

Soln.

(i) If $f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$, then by formula V above,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2} \, ds = \frac{1}{2} \int_s^\infty \frac{2s}{(s^2+a^2)^2} \, ds = -\frac{1}{2} \left(\frac{1}{s^2+a^2}\right)_s^\infty = \frac{1}{2} \cdot \frac{1}{s^2+a^2}$$

$$\therefore \frac{f(t)}{t} = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{2a}$$

$$\text{Hence, } f(t) = \frac{1}{2a} t \sin at$$

$$\text{Otherwise: Let } f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} \text{ so that } f(s) = \frac{1}{s^2+a^2}$$

They by (IV) above, $t f(t) = L^{-1} \left\{ -\frac{d}{ds} [\bar{f}(s)] \right\} = L^{-1} \left\{ -\frac{d}{ds} \left(\frac{1}{s^2 + a^2} \right) \right\}$

or $\frac{t \sin at}{a} = L^{-1} \left\{ \frac{2s}{(s^2 + a^2)^2} \right\}$. Hence $L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at$

(ii) In (i), we have proved that,

$$L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$L^{-1} \left\{ \frac{s^2}{(s^2 + a^2)^2} \right\} = L^{-1} \left\{ s \cdot \frac{s}{(s^2 + a^2)^2} \right\} = \frac{d}{dt} [f(t)]$$

$$= \frac{d}{dt} \left(\frac{1}{2a} t \sin at \right) = \frac{1}{2a} (\sin at + at \cos at)$$

(iii) In (i), we have shown that,

$$L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} (t \sin at) = f(t), \text{ say}$$

By formula III above, we have,

$$L^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} = L^{-1} \left\{ \frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2} \right\} = \int_0^t f(t) dt = \int_0^t \frac{t \sin at}{2a} dt$$

$$= \frac{1}{2a} \left[\left. t \cdot \frac{-\cos at}{a} \right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a} \right) dt \right]$$

$$= \frac{1}{2a} \left[\frac{-t \cos at}{a} + \frac{\sin t}{a^2} \right] = \frac{1}{2a^3} (\sin at - at \cos at)$$

Example 22 Find the invers Laplace transform of

(i) $\frac{s+2}{s^2(s+1)(s-2)}$ (ii) $\frac{s+2}{(s^2+4s+5)^2}$

Soln. (i) $L^{-1} \left\{ \frac{s+2}{(s+1)(s-2)} \right\} = \frac{4}{3} L^{-1} \left(\frac{1}{s-2} \right) - \frac{1}{3} L^{-1} \left(\frac{1}{s+1} \right) = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}$

By III above, $L^{-1} \left\{ \frac{s+2}{s(s+1)(s-2)} \right\} = \int_0^t L^{-1} \left\{ \frac{s+2}{(s+1)(s-2)} \right\} dt$

$$= \int_0^t \left(\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} \right) dt = \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1$$

Again by III above, $L^{-1} \frac{s+2}{s^2(s+1)(s-2)} = \int_0^t L^{-1} \left\{ \frac{s+2}{s(s+1)(s-2)} \right\} dt$

$$= \int_0^t \left\{ \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1 \right\} dt = \frac{1}{3} (e^{2t} - e^{-t} - t)$$

(ii) $L^{-1} \left(\frac{1}{s^2+4s+5} \right) = L^{-1} \left\{ \frac{1}{(s+2)^2+1} \right\} = e^{-2t} \sin t$

By II above, $L^{-1} \left\{ \frac{d}{ds} \left(\frac{1}{s^2+4s+5} \right) \right\} = (-1)^1 t e^{-2t} \sin t$

i.e.

$$L^{-1} \left\{ \frac{-(2s+4)}{(s^2+4s+5)^2} \right\} = -t e^{-2t} \sin t$$

or

$$L^{-1} \left\{ \frac{s+2}{(s^2+4s+5)^2} \right\} = \frac{1}{2} t e^{-2t} \sin t$$

Example 23 Find the inverse Laplace transforms of the following

(i) $\log \frac{s+1}{s-1}$

(ii) $\cot^{-1} \left(\frac{s}{2} \right)$

(iii) $\tan^{-1} \left(\frac{2}{s^2} \right)$

Soln. (i) If $f(t) = L^{-1} \log \frac{s+1}{s-1}$, then by IV above,

$$\begin{aligned} t f(t) &= L^{-1} \left\{ -\frac{d}{ds} \log \left(\frac{s+1}{s-1} \right) \right\} = L^{-1} \left\{ \frac{d}{ds} \log(s+1) \right\} + L^{-1} \left\{ \frac{d}{ds} \log(s-1) \right\} \\ &= -L^{-1} \left(\frac{1}{s+1} \right) + L^{-1} \left(\frac{1}{s-1} \right) = e^{-t} + e^t = 2 \sinh t \end{aligned}$$

Thus $f(t) = (2 \sinh t) / t$

(ii) If $f(t) = L^{-1} \cot^{-1} \left(\frac{s}{2} \right)$, then by IV above,

$$t f(t) = L^{-1} \left\{ -\frac{d}{ds} \cot^{-1} \left(\frac{s}{2} \right) \right\} = L^{-1} \left(\frac{2}{s^2 + 2^2} \right) = \sin 2t$$

Thus, $f(t) = (\sin 2t) / t$

(iii) If $f(t) = L^{-1} \left(\tan^{-1} \frac{2}{s^2} \right)$, then by IV above,

$$\begin{aligned} t f(t) &= L^{-1} \left\{ -\frac{d}{ds} \tan^{-1} \left(\frac{2}{s^2} \right) \right\} = L^{-1} \left\{ \frac{4s}{s^4 + 4} \right\} \\ &= L^{-1} \left\{ \frac{4s}{(s^2 + 2) - (2s)^2} \right\} = L^{-1} \left\{ \frac{4s}{(s^2 + 2 + 2s)(s^2 + 2 - 2s)} \right\} \\ &= e^t \sin t - e^{-t} \sin t = 2 \sinh t \sin t \end{aligned}$$

4.14 CONVOLUTION THEOREM OF LAPLACE TRANSFORM

If $L^{-1} \{ \bar{f}(s) \} = f(t)$ and $L^{-1} \{ \bar{g}(s) \} = g(t)$,

then $L^{-1} \{ \bar{f}(s) \bar{g}(s) \} = \int_0^t f(u) g(t-u) du = F * G$

[$F * G$ is called the convolution or falting of F and G].

Let $\phi(t) = \int_0^t f(u) g(t-u) du$

$$L\phi(t) = \int_0^\infty e^{-st} \left\{ \int_0^t f(u) g(t-u) du \right\} dt \quad (1)$$

$$= \int_0^\infty \int_0^t e^{-st} f(u) g(t-u) du dt$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$.

On changing the order of integration, we get,

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u) dt du$$

$$= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-s(t-u)} g(t-u) dt \right\} du$$

$$= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-sv} g(v) dv \right\} dv \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u) g(s) du = \int_0^\infty e^{-su} f(u) du \cdot \bar{g}(s)$$

$$= \bar{f}(s) \cdot \bar{g}(s) \text{ whence follows the desired result.}$$

Example.24

Apply Convolution theorem to evaluate $L^{-1} \frac{s}{(s^2 + a^2)^2}$

Soln.

Since $f(t) = L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at$ and $g(t) = L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{1}{a} \sin at$

$\therefore$ by Convolution theorem, we get

$$L^{-1} \left\{ \frac{s}{s^2 + a^2} \cdot \frac{1}{s^2 + a^2} \right\} = \int_0^t \cos au \frac{\sin a(t-u)}{a} du$$

$$= \frac{1}{2a} \int_0^t [\sin at - \sin(2au - at)] dt$$

$$\left[\begin{aligned} \because f(u) &= \cos at \\ g(t-u) &= \frac{1}{a} \sin a(t-u) \end{aligned} \right]$$

$$= \frac{1}{2a} \left[u \sin at + \frac{1}{2a} \cos(2au - at) \right]_0^t = \frac{1}{2a} t \sin at$$

Hence $L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at$

EXERCISE 4.5 :

Find the inverse transform of

1. $\frac{1}{s^2(s+5)}$

2. $\frac{s}{a^2 s^2 + b^2}$

3. $\frac{1}{s^2(s^2 + a^2)}$

4. $\frac{1}{s^2(s^2 + 1)}$

5. $\frac{2as}{(s^2 + a^2)^2}$

6. $\log \left(\frac{1+s}{s} \right)$

7. $\log \left(\frac{s+a}{s+b} \right)$

8. $L^{-1} \log \left\{ \frac{s+1}{(s+2)(s+3)} \right\}$

Using Convolution theorem, evaluate :

9. $L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\}$

10. $L^{-1} \left\{ \frac{1}{s(s^2 + 4)} \right\}$

11. $L^{-1} \frac{1}{s^2(s^2 + a^2)}$

ANSWERS 4.5

1. $\frac{1}{25} [e^{-5t} + 5t - 1]$

2. $\frac{1}{a^2} \cos \left(\frac{bt}{a} \right)$

3. $\frac{1}{a^2} \left[t - \frac{1}{a} \sin at \right]$

4. $\frac{1}{t^2} + \cos t + 1$

5. $t \sin t$

6. $\left(\frac{1-e^t}{t} \right)$

7. $\frac{1}{t} [e^{-bt} - e^{-at}]$

8. $e^{-t} - e^{-2t} - e^{-3t}$

9. $\frac{e^{-bt} - e^{-at}}{a-b}$

10. $\frac{1}{4} [1 - \cos 2t]$

11. $\frac{1}{a^3} (at - \sin at)$

MULTIPLE CHOICE QUESTIONS

1. Laplace transform of $(t \sin t)$ is,....

(a) $\frac{2s}{(s^2 + 1)^2}$

(b) $\frac{2s}{(s+1)^2}$

(c) $\frac{2s}{(s+1)}$

(d) $\frac{2s}{(s^2 + 1)}$

2. Laplace transform of $f(t)t \geq 0$ is defined as

(a) $\int_0^\infty e^{-st} f(t) dt$

(b) $\int_0^\infty e^{-st} f(t) dt$

(c) $\int_0^\infty e^{-st} f(t) dt$

(d) $\int_0^\infty e^{-t} f(t) dt$

3. A unit step functions is defined as

(a) $u(t+a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

(b) $u(ta) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

(c) $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

(d) $u(a-t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

4. $L(e^{2t} \sin t) \dots\dots$

(a) $\frac{1}{s^2 + 4s + 5}$

(b) $\frac{1}{s^2 - 4s - 5}$

(c) $\frac{1}{s^2 - 4s}$

(d) $\frac{1}{s^2 - 4s + 5}$

5. Inverse Laplace transform of $(s+2)^{-2}$ is.....

(a) $t e^{-2t}$

(b) $t - e^{-2t}$

(c) $t e^{2t}$

(d) $t + e^{-2t}$

6. Inverse Laplace transform of $1/(s^2 + 4s + 13)$ is.....

(a) $\frac{1}{3} e^{-2t} \sin t$

(b) $\frac{1}{3} e^{-2t} \sin 3t$

(c) $e^{-2t} \sin 3t$

(d) $\frac{1}{3} e^{-t} \sin 3t$

LAPLACE TRANSFORMS / 186

LAPLACE TRANSFORMS / 187

19. $L^{-1} = (1/s^n)$ is possible only when in is
 (a) zero (b) -ve integer
 (c) +ve integer (d) negative rational
20. $L^{-1}[\phi(s)] = f(t)$, then $L^{-1}[e^{-as} \cdot \phi(s)] =$
 (a) $f(t+a) \cdot u(t-a)$ (b) $f(t-a) \cdot u(ta)$
 (c) $t(t-a) \cdot u(t+a)$ (d) $f(t-a) \cdot u(t-a)$
21. Laplace transform of the unit impulse function $\delta(t-a) = \dots$
 (a) e^{-as} (b) e^{as}
 (c) e^{-s} (d) e^{-a}
22. $L^{-1}\left\{\frac{s^2+3s+7}{s^3}\right\} = \dots$
 (a) $1-3t+7\frac{t^2}{2}$ (b) $1+3t+7\frac{t^2}{2}$
 (c) $1+3t-7\frac{t^2}{2}$ (d) $1-3t-7\frac{t^2}{2}$
23. If $L\{f(t)\} = \bar{f}(s)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \dots$
 (a) $\int_0^t f(t) dt$ (b) $\int_0^t f(s) dt$
 (c) $\int_0^t f(t) dt$ (d) $\int_0^t f(t) ds$
24. If $f(t)$ is a periodic function with period T, then $L\{f(t)\}$
 (a) $\int_0^T \frac{e^{-t} \cdot f(t) dt}{(1-e^{-st})}$ (b) $\int_0^T \frac{e^{st} \cdot f(t) dt}{(1-e^{st})}$
 (c) $\int_0^T \frac{e^{-st} \cdot f(t) dt}{(1-e)} (d) \int_0^T \frac{e^{-st} \cdot f(t) dt}{(1-e^{-st})}$
25. If y satisfies $y'' + 3y' + 2y = e^{-t}$ with $y(0) = y'(0) = 0$, then $L\{y(t)\} = \dots$
 (a) $\frac{1}{(s+1)^2(s+2)}$ (b) $\frac{1}{(s+1)(s+2)}$
 (c) $\frac{1}{(s-1)^2(s-2)}$ (d) $\frac{1}{(s-1)(s+2)}$

26. $L[e^{3t}(2\cos 5t + 3\sin 4t)] = \dots$
 (a) $\frac{2}{s^2-6s+34} + \frac{12}{s^2-6s+25}$ (b) $\frac{2(s-3)}{s^2-6s+34} + \frac{12}{s^2-6s+25}$
 (c) $\frac{(s-3)}{s^2-6s+34} + \frac{12}{s^2-6s+25}$ (d) $\frac{6}{s^2-6s+34} + \frac{12}{s^2-6s+25}$
27. $L(4^t) = \dots$
 (a) $\frac{1}{(\log 4)}$ (b) $\frac{1}{(s+\log 4)}$
 (c) $\frac{1}{(s-\log 4)}$ (d) $\frac{1}{(s-\log 4)^2}$
28. $L^{-1}\left\{\frac{1}{\sqrt{s+3}}\right\} = \dots$
 (a) $\frac{e^{3t}}{\sqrt{\pi t}}$ (b) $\frac{e^{-t}}{\sqrt{\pi t}}$
 (c) $\frac{e^{-3t}}{\sqrt{\pi}}$ (d) $\frac{e^{-3t}}{\sqrt{\pi t}}$
29. $\frac{d^2}{ds^2}\{Lf(t)\} - L[t^2 f(t)] = 0$
 (a) True (b) False
 (c) zero (d) none
30. If $L\{f(t)\} = \phi(s)$, then $L\{tf(t)\} = \frac{d}{ds}[\phi(s)]$
 (a) True (b) False
 (c) zero (d) none
31. $L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = \dots$
 (a) e^{at} (b) e^{-at}
 (c) te^{-at} (d) t
32. Laplace transform of $t^4 e^{-at}$ is
 (a) $\frac{4!}{(s+a)^4}$ (b) $\frac{5!}{(s-a)^5}$

33. $L^{-1} \frac{s^2}{(s^2+4)^2}$ is
- (a) $\frac{1}{4} \sin 2t + \frac{1}{2} \cos 2t$
 (c) $\sin 2t + \frac{t}{2} \cos 2t$
34. $L^{-1} \frac{1}{s(s^2+1)}$ is
- (a) $1 + \sin t$
 (c) $1 + \cos t$
35. $L[\sin 2t \sin 3t]$
- (a) $\frac{12s}{(s^2-1)(s^2-2s)}$
 (c) $\frac{12s}{(s^2+1)(s^2+25)}$
36. Laplace transform of $\int_0^t t^3 dt = \dots$
- (a) $\frac{4}{s^2}$
 (c) $\frac{4}{s^4}$
37. Laplace transform of $\int_0^t e^{-3t} dt =$
- (a) $\frac{1}{s(s+3)}$
 (c) $\frac{1}{(s+3)s^2}$
- (d) $\frac{5!}{(s+a)^5}$
 (b) $\frac{1}{4} \sin 2t + \cos 2t$
 (d) $\frac{1}{2} \sin 2t + \frac{t}{4} \cos 2t$
 (b) $1 - \cos t$
 (d) $1 - s \sin t$
 (b) $\frac{12s}{(s+1)(s+2s)}$
 (d) None
 (b) $\frac{2}{s^2}$
 (d) $\frac{2}{s^4}$
 (b) $\frac{1}{s(s+4)}$
 (d) None

38. Laplace transform of $t \cos(at+b) =$
- (a) $\frac{(s^2+a^2) \cos b + 2s \sin b}{(s^2+a^2)^2}$
 (b) $\frac{(s^2-a^2) \cos b + 2s \sin b}{(s^2+a^2)^2}$
 (d) None
39. Laplace transform of $t^2 \cos(4t+7) =$
- (a) $\frac{4}{(s+2)^3}$
 (b) $\frac{4}{(s-2)^3}$
 (c) $\frac{2}{(s+2)^3}$
 (d) $\frac{2}{(s+4)^2}$
40. Laplace transform of $\int_0^t \frac{\sin x}{x} dx =$
- (a) $\frac{1}{s} \tan^{-1} s$
 (b) $\cot^{-1}(s)$
 (c) $\tan^{-1} s$
 (d) $\frac{1}{s} \cot^{-1}(s)$
41. $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$ then $L[f(t)] =$
- (a) $e^{-\frac{2\pi}{3}s} \frac{s}{s^2+1}$
 (b) $e^{\frac{2\pi}{3}s} \frac{s}{s^2+1}$
 (c) $e^{-\frac{2\pi}{3}s} \frac{1}{s^2+1}$
 (d) None
42. Find the Laplace transform of $|\sin x|$ is
- (a) $\frac{1}{s+1}$
 (b) $\frac{1}{s^2+1}$
 (c) $\frac{s}{s^2+1}$
 (d) $\frac{s}{s+1}$

43. Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$
- (a) $\frac{s}{s^2+1}$ (b) $\frac{1}{s+1}$
 (c) does not exist (d) 0
44. Laplace transform of $\cos(\omega t + \theta)$ is
- (a) $\frac{s \cos \theta + w \sin \theta}{s^2 + w^2}$ (b) 0
 (c) $\frac{\cos \theta - \sin \theta}{s^2 + w^2}$ (d) $\frac{s \cos \theta - w \sin \theta}{s^2 + w^2}$
45. If $L[f(t)] = \frac{2s-1}{s(s-1)}$ then $L\left[f(2t) - f\left(\frac{t}{2}\right)\right] = \frac{k}{(s-2)(2s-1)}$ then $k =$
- (a) $k=3$ (b) $k=0$
 (c) $k=2$ (d) $k=1$
46. $L^{-1}[\cot^{-1}(1+s)] =$
- (a) $\frac{1}{t} e^{-t} \cos t$ (b) $\frac{1}{t} e^{-t} \sin t$
 (c) $t e^{-t} \sin t$ (d) $t \cos^{-1} t$
47. Inverse of Laplace transform $\tan^{-1}\left(\frac{2}{s}\right)$ is,
- (a) $\frac{\cos 2t}{t}$ (b) $\frac{\sin 2t}{t^2}$
 (c) $\frac{\sin 2t}{t}$ (d) $\frac{\cos 2t}{t^2}$
48. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F(ks)] =$
- (a) $k f(t/k)$ (b) $f\left(\frac{k}{t}\right)$
 (c) $f\left(\frac{t}{k}\right)$ (d) $\frac{1}{k} f\left(\frac{t}{k}\right)$

49. If $f(s) = \ln\left(1 + \frac{a^2}{s^2}\right)$ and $L^{-1}[F(s)] = f(t)$ then $f(t) =$
- (a) $\frac{2}{t}(1 - \cos at)$ (b) $\frac{2}{t}(1 + \cos at)$
 (c) $\frac{2}{t}(1 + \cos at)$ (d) None
50. $L^{-1}\left[\frac{6s-4}{s^2-4s+20}\right] =$
- (a) $e^t [6 \cos 4t + 2 \sin 4t]$ (b) $e^{2t} [6 \cos 4t + 2 \sin 4t]$
 (c) $e^{2t} [6 \cos 4t - 2 \sin 4t]$ (d) None
51. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F(s-a)] =$
- (a) $e^t f(t)$ (b) $e^{at} f(t)$
 (c) $e^{at} f(t)$ (d) None

ANSWER KEY (MULTIPLE CHOICE QUESTIONS):

01.(a)	02.(b)	03. (c)	04. (d)	05. (a)	06. (b)
07.(c)	08.(d)	09. (a)	10. (b)	11. (c)	12. (d)
13.(a)	14.(b)	15. (c)	16. (d)	17. (a)	18. (b)
19.(c)	20.(d)	21. (a)	22. (b)	23. (c)	24. (d)
25.(a)	26.(b)	27. (c)	28. (d)	29. (a)	30. (b)
31.(c)	32.(d)	33. (a)	34. (b)	35. (c)	36. (d)
37.(a)	38.(b)	39. (c)	40. (d)	41. (a)	42. (b)
43.(c)	44.(d)	45. (a)	46. (b)	47. (c)	48. (d)
49.(a)	50.(b)	51. (c)			

5

FOURIER TRANSFORMS

5.1 INTRODUCTION :

In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ends with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

5.2 DEFINITION

The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x) dx$$

where $K(s, x)$ is called the kernel of the transform and is a known function of s and x function $f(x)$ is called the inverse transform of $\bar{f}(s)$.

Three simple examples of a kernel are as follows,

(i) When $K(s, x) = e^{-sx}$, it leads to the Laplace transform of $f(x)$, i.e.

$$\bar{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx$$

(ii) When $K(s, x) = e^{isx}$, we have the Fourier transform of $f(x)$, i.e.

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

(iii) When $K(s, x) = x^{s-1}$, it gives the Mellin transform of $f(x)$ i.e.

$$M(s) = \int_0^{\infty} f(x) x^{s-1} dx$$

5.3 (1) Fourier Integral Theorem

Consider a function $f(x)$ which satisfies the Dirichlet's conditions in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad (1)$$

where $a_0 = \frac{1}{c} \int_{-c}^c f(t) dt$, $a_n = \frac{1}{c} \int_{-c}^c f(t) \cos \frac{n\pi t}{c} dt$, and $b_n = \frac{1}{c} \int_{-c}^c f(t) \sin \frac{n\pi t}{c} dt$

Substituting the values of a_0 , a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c} \int_{-c}^c f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} f(t) \cos \frac{n\pi(t-x)}{c} dt \quad (2)$$

If we assume that $\int_{-c}^c |f(x)| dx$ converges, the first term on the right side of (2) approaches 0 as $c \rightarrow \infty$, since

$$\left| \frac{1}{2} \int_{-c}^c f(t) dt \right| \leq \frac{1}{2c} \int_{-c}^c |f(t)| dt$$

The second term on the right side of (2) tends to

$$\begin{aligned} & \lim_{c \rightarrow \infty} \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos \frac{n\pi(t-x)}{c} dt \\ &= \lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda(t-x) dt, \text{ on writing } \pi/c = \delta\lambda \end{aligned}$$

This is of the form $\lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} f(n\delta\lambda)$, i.e. $\int_0^{\infty} F(\lambda) d\lambda$

Thus as $c \rightarrow \infty$ (2) becomes $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda$

(3)

Which is known as the Fourier integral of $f(x)$

(2) Fourier sine and cosine integrals.

Expanding $\cos \lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^\infty f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_{-\infty}^\infty f(t) \sin \lambda t dt d\lambda \quad (4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t dt d\lambda \quad (5)$$

which is known as the Fourier sine integral.

Similarly, if $f(x)$ is an even function, (4) take the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t dt d\lambda \quad (6)$$

Which is known as the Fourier cosine integral.

(3) Complex form of Fourier integrals. Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t-x) dt d\lambda \quad (7)$$

because $\cos \lambda(t-x)$ is an even function of λ . Also since $\sin \lambda(t-x)$ is an odd function of λ , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \lambda(t-x) dt d\lambda \quad (8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) e^{i\lambda(t-x)} dt d\lambda \quad (9)$$

Which is the complex form of the Fourier integral.

(4) Fourier integral representation of a function.

Using (4), a function $f(x)$ may be represented by a Fourier integral as

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] d\lambda \quad (10)$$

$$\text{where } A(\lambda) = \int_{-\infty}^\infty f(t) \cos \lambda t dt; B(\lambda) = \int_{-\infty}^\infty f(t) \sin \lambda t dt$$

If $f(x)$ is an odd function, then by [by (5)] (11)

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin \lambda x d\lambda \text{ where } B(\lambda) = 2 \int_0^\infty f(t) \sin \lambda t dt$$

If $f(x)$ is an even function, then [by (6)] (12)

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x d\lambda \text{ where } A(\lambda) = 2 \int_0^\infty f(t) \cos \lambda t dt$$

Example .1 Express $f(x) = 1$ for $0 \leq x \leq \pi$,
 $= 0$ for $x > \pi$,

as a Fourier sine integral and hence evaluate. $\int_0^\infty \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(\lambda x) d\lambda$

Soln. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\infty f(t) \sin(\lambda t) dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\pi \sin(\lambda t) dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \left[\frac{-\cos(\lambda t)}{\lambda} \right]_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) \begin{cases} \pi/2 & \text{for } 0 \leq x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2} \left[\frac{f(\pi-0) + f(\pi+0)}{2} \right] = \frac{\pi}{2} \cdot \frac{1+0}{2} = \frac{\pi}{4}$$

5.4 (1) Fourier Transforms

Bewriting (9) of 3 as $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} dx \int_{-\infty}^{\infty} f(t) e^{ist} dt$,

it Follows that if

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{ist} dt \quad (1)$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{isx} ds \quad (2)$$

The function $F(s)$, defined by (1), is called the Fourier transform of $f(x)$. Also the function as given (2), is called the inverse. Fourier transform of $F(s)$,

(2) **Fourier sine and cosine transforms.** From (5) of 3, it follows that if

$$F(s) = \int_0^{\infty} f(x) \sin sx \, dx \quad (3)$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(s) \sin sx \, ds \quad (4)$$

The function $F_s(s)$, as defined by (3), is known as the Fourier sine transform of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (4) is called the inverse Fourier sine transform of $F_s(s)$. Similarly it follows from (6) of 3 that if

$$F_c(s) = \int_0^{\infty} f(x) \cos sx \, dx \quad (5)$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx \, ds \quad (6)$$

The function $F_c(s)$ as defined by (5) is known as the Fourier cosine transforms of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (6), is called the inverse Fourier cosine transform of $F_c(s)$.

3)(a) **Finite Fourier sine and cosie transforms.** The finite Fourier sine transforms of $f(x)$, in $0 < x < c$, is defined as

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c} dx, \text{ where } n \text{ in an integer.}$$

The function $f(x)$ is then called the inverse finite Fourier sine transform of $F_s(n)$

$$\text{is given by. } f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{c}$$

3)(b) **Finite Fourier cosie transforms.** The finite Fourier cosine transforms of $f(x)$, in $0 < x < c$, is defined as

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c} dx, \text{ where } n \text{ is an integer,}$$

The function $f(x)$ is the called as invers finite Fourier cosine transforms of which is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c}$$

5.5

PROPERTIES OF FOURIER TRANSFORMS

(1)

Linear property : if $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively then

$$F[af(x) + bg(x)] = aF(s) + bG(s)$$

where a and b are constants,

we have,

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx \text{ and } G(s) = \int_{-\infty}^{\infty} e^{isx} g(x) dx$$

$\therefore$

$$F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)] dx = a \int_{-\infty}^{\infty} e^{isx} f(x) dx + b \int_{-\infty}^{\infty} e^{isx} g(x) dx$$

$$= aF(s) + bG(s)$$

(2)

Change of scale property : If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F(f(ax)) = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

we have

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$F[f(ax)] = \int_{-\infty}^{\infty} e^{isx} f(ax) dx$$

$$\left| \begin{array}{l} \text{Put } ax = t \\ \text{so that } dx = dt/a \end{array} \right.$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t) dt/a = \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt = \frac{1}{a} F\left(\frac{s}{a}\right) \quad [\text{by (i)}]$$

Corr. If $F_s(s)$ and F_c are the Fourier sine and cosine transforms of $f(x)$ respectively,

$$\text{then } F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right) \text{ and } F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

(3) **Shifting Property:** If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(x-a)] = e^{isa} F(s)$$

we have

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$\left| \begin{array}{l} \text{Put } x-a = t \\ \text{so that } dx = dt \end{array} \right.$$

$$F[f(x-a)] = \int_{-\infty}^{\infty} e^{ist} f(t) dt$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t) dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t) dt = e^{isa} F(s) \quad [\text{by (i)}]$$

(4) **Modulation theorem:** If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F(f(x) \cos ax) = \frac{1}{2} \{F(s+a) + F(s-a)\}$$

we have

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$\therefore F[f(x) \cos ax] = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax dx = \int_{-\infty}^{\infty} e^{isx} f(x) \cdot \frac{e^{iax} + e^{-iax}}{2} dx$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx \right] = \frac{1}{2} [F(s+a) + F(s-a)]$$

Cor. If $f_s(s)$ and $F_c(s)$ are Fourier sine and cosine transform of $f(x)$ respectively,

then

$$(i) \quad F_s[f(x) \cos ax] = \frac{1}{2} [F_s(s+a) + F_s(s-a)]$$

$$(ii) \quad F_c[f(x) \sin ax] = \frac{1}{2} [F_s(s+a) + F_s(s-a)]$$

$$(iii) \quad F_s[f(x) \sin ax] = \frac{1}{2} [F_c(s-a) - F_c(s+a)]$$

8195

Example.2 Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases} \quad \text{Hence } \int_0^{\infty} \frac{\sin x}{x} dx$$

Soln. The Fourier transform of $f(x)$, i.e.

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-1}^1 (1) e^{isx} dx = \left[\frac{e^{isx}}{is} \right]_{-1}^1 = \frac{e^{is} - e^{-is}}{is}$$

Thus $F[f(x)] = F(s) = 2 \frac{\sin s}{s}$, $s \neq 0$. For $s=0$ we have $F(s)=2$

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{isx} ds, \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Putting $x=0$, we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \quad \therefore \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example.3

Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^2} \cos \frac{x}{2} dx$

Soln. $F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx} dx = F(s)$, say

$$= \int_{-\infty}^{-1} (0)e^{isx} dx + \int_{-1}^1 (1-x^2)e^{isx} dx + \int_1^{\infty} (0)e^{isx} dx = \left[(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3} \right]_{-1}^1$$

$$= 2\left(\frac{e^{is} + e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is} - e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s \cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$

$$\text{or } -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3}(s \cos s - \sin s)e^{-isx} ds = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Putting $x = 1/2$ we obtain

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3}(s \cos s - \sin s)e^{-is/2} ds = \frac{3}{4}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \frac{s}{2} - i \sin \frac{s}{2} \right) ds = \frac{-3\pi}{8}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \cdot \cos \frac{s}{2} ds = -\frac{3\pi}{8}$$

$$\text{or } \int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cdot \cos \frac{x}{2} dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example.4 (a) Find the Fourier transforms of $e^{-a^2 x^2}$, $a > 0$. Hence deduce that $e^{-x^2/2}$ self reciprocal in respect of Fourier transforms.

(b) Find Fourier transform of (i) $e^{-2(x-3)^2}$ (ii) $e^{-x^2} \cos 3x$

Soln. (a) $F(e^{-a^2 x^2}) = \int_{-\infty}^{\infty} e^{-a^2 x^2} \cdot e^{isx} dx = \int_{-\infty}^{\infty} e^{-a^2 x^2 (x^2 - isx/a^2)} dx$

$$= \int_{-\infty}^{\infty} e^{-a^2 (x - is/2a^2)^2} \cdot e^{-s^2/4a^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2} dt / a$$

$$\left[\text{Putting } a(x - is/2a^2) = t, dx = dt/a \right]$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi}$$

$$\left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \right]$$

$$\text{Hence } F(e^{-a^2 x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

The Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are same. Hence it follows that $e^{-x^2/2}$ is self-reciprocal under the Fourier transform.

(b) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

by change of scale property, $F[f(2x)] = \frac{1}{2} F(s/2)$

$$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F[f(x-3)] = e^{i3s} F(s)$

$$F(e^{-2(x-3)^2}) = e^{i3s} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)}$$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]$$

(ii)

Example .5 Find the Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

Soln. Fourier cosine transform of $f(x)$ i.e. $F_c[f(x)]$

$$\begin{aligned} &= \int_0^\infty f_c(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^\infty 0 \, dx \\ &= \left[x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right]_0^1 + \left[(2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right]_1^2 \\ &= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} \right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right) \\ &= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2} \end{aligned}$$

Example .6 Find the Fourier sine transforms of $e^{-|x|}$

Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi e^{-m}}{2}, m > 0$

Soln. x being positive in the interval $(0, \infty), e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$\begin{aligned} F_s\{f(x)\} &= \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-x} \sin sx \, dx \\ &= \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty = \frac{s}{1+s^2} \end{aligned}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s[f(x)] \sin sx \, ds \text{ or } e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

$$\text{or changing } x \text{ to } m, e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s \sin ms}{1+s^2} \, ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+m^2} \, dx$$

Hence $\int_0^\infty \frac{x \sin mx}{1+m^2} \, dx = \frac{\pi e^{-m}}{2}$

Example.7 Find the Fourier cosine transform of e^{-x^2}

Soln. We have $F_c(e^{-x^2}) = \int_0^\infty e^{-x^2} \cos sx \, dx = I$ (say)

Differentiating under the integral sign w.r.t. s .

$$\begin{aligned} \frac{dI}{ds} &= -\int_0^\infty x e^{-x^2} \sin sx \, dx = \frac{1}{2} \int_0^\infty (\sin sx) (-2xe^{-x^2}) \, dx \\ &= \frac{1}{2} \left\{ \left[\sin sx \cdot e^{-x^2} \right]_0^\infty - \int_0^\infty \cos sx \cdot e^{-x^2} \, dx \right\} \\ &= -\frac{s}{2} \int_0^\infty e^{-x^2} \cos sx \, dx = -\frac{s}{2} I \text{ or } \frac{dI}{I} = -\frac{s}{2} ds + \log c \end{aligned}$$

$$\log I = \frac{-s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$$I = ce^{-s^2/4} \text{ or } \int_0^\infty e^{-x^2} \cos sx \, dx = ce^{-s^2/4}$$

$$\text{Putting } s=0, c = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \text{ i.e. } I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

$$\text{Hence } F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

Example .8 Find the Fourier sine transforms of e^{-ax}/x

Soln. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform.

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t s , we get

$$\frac{d}{ds} [F(s)] = \int_0^\infty \frac{x e^{-ax} \cos sx}{x} \, dx = \int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t s , we obtain $F(s) = \int \frac{a}{s^2 + a^2} ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0, \therefore c = 0$ Hence $F(s) = \tan^{-1}(s/a)$

Example.9 Find the Fourier cosine transform of $f(x) = 1/(1+x^2)$

Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$

Soln.

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I, \text{ say}$$

$$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin x}{1+x^2} dx = - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx$$

$$= - \int_0^\infty \frac{[(1+x^2)-1] \sin sx}{x(1+x^2)} dx = - \int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx$$

$$\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = I$$

$$\frac{d^2 I}{ds^2} - I = 0 \text{ or } (D^2 - 1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $I = c_1 e^s + c_2 e^{-s}$

$$dI/ds = c_1 e^s = c_2 e^{-s}$$

When $s = 0$, (i) and (ii) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (iii) and (iv) give $c_1 - c_2 = -\pi/2$

Solving these, $c_1 = 0, c_2 = \pi/2$

Thus from (i) and (iv), we have $F_c\{f(x)\} = I = (\pi/2)e^{-s}$

Now

$$F_s[\phi(x)] = \int_0^\infty \frac{x \sin x}{1+x^2} dx = \frac{dI}{ds}, \text{ from (ii)}$$

$$= (\pi/2)e^{-s}, \text{ from (v), with } c_1 = 0, c_2 = \pi/2$$

Example.10 Find the Fourier sine and cosine transform of $x^{n-1}, n > 0$.

Soln. We know that $f_s(x^{n-1}) = \int_0^\infty x^{n-1} \sin sx dx$

$$f_c(x^{n-1}) = \int_0^\infty x^{n-1} \cos sx dx$$

$$\therefore F_c(x^{n-1}) + iF_s(x^{n-1}) = \int_0^\infty (\cos sx + i \sin sx) x^{n-1} dx$$

$$= \int_0^\infty e^{its} x^{n-1} dx = \int_0^\infty e^{-t} \left(-\frac{t}{is}\right)^{n-1} \left(-\frac{dt}{is}\right)$$

$$= \left(\frac{-1}{i}\right)^n \int_0^\infty e^{-t} t^{n-1} dt = \frac{(i)^{2n}}{(i)^n s^n} \Gamma(n) = \frac{(i)^n}{s^n} \Gamma(n)$$

$$= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^n \Gamma(n) / s^n = \left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right) \Gamma(n) / s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2}$$

and

$$F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}$$

Example.11

(a) Show that $F_s[xf(x)] = -\frac{d}{ds}\{F_c(s)\}; F_c[xf(x)] = \frac{d}{ds}\{F_s(s)\}$

(b) Find the Fourier sine and cosine transform of xe^{-ax}

Soln. (a)

$$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x) \cos sx dx\right\} = \int_0^\infty f(x) (-x \sin sx) dx$$

$$= -\int_0^\infty \{xf(x)\} \sin sxdx = -F_s[xf(x)]$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x) \sin sxdx\right\} = \int_0^\infty f(x) (x \cos sx) dx$$

$$= \int_0^\infty \{xf(x)\} \cos sxdx = F_c\{xf(x)\}$$

(b) We have

$$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sxdx = \frac{e^{-ax}}{a^2 + s^2} \Big|_0^\infty - a \sin sx - s \cos sx \Big|_0^\infty$$

$$= \frac{s}{a^2 + s^2}$$

and

$$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sxdx = \frac{e^{-ax}}{a^2 + s^2} \Big|_0^\infty - a \cos sx + s \sin sx \Big|_0^\infty$$

$$= \frac{a}{a^2 + s^2} \quad \text{Now } F_s(xe^{-ax}) = \frac{d}{ds}\{F_c(e^{-ax})\}$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2 + s^2}\right) = \frac{2as}{(a^2 + s^2)^2}$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\}$$

$$= \frac{d}{ds}\left(\frac{s}{a^2 + s^2}\right) = \frac{(a^2 + s^2) - s(2s)}{(a^2 + s^2)^2} = \frac{a^2 - s^2}{(a^2 + s^2)^2}$$

Example.12 If the Fourier sine transform of $f(x) = \frac{1 - \cos n\pi}{n^2 \pi^2} (0 \leq x \leq \pi)$. Find $f(x)$

Soln. We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi} \sum_{n=1}^\infty F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi} \sum_{n=1}^\infty \left\{ \frac{1 - \cos n\pi}{n^2 \pi^2} \right\} \sin nx$$

$$= \frac{2}{\pi^3} \sum_{n=1}^\infty \left\{ \frac{1 - \cos nx}{n^2} \right\} \sin nx.$$

Example.13 Solve the integral equation.

$$\int_0^\infty f(\theta) \cos \alpha \theta d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}, \text{ Hence evaluate } \int_0^\infty \frac{\sin^2 t}{t^2} dt$$

Soln. We have $\int_0^\infty f(\theta) \cos \alpha \theta d\theta = F_c(\alpha)$

$$\therefore F_c(\alpha) = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases} \quad (i)$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \int_0^1 (1 - \alpha) \cos \alpha \theta d\alpha \quad [\text{Integrating by parts}]$$

$$= \frac{2}{\pi} \left[(1 - \alpha) \frac{\sin \alpha \theta}{\theta} \Big|_0^1 - \int_0^1 (-1) \frac{\sin \alpha \theta}{\theta} d\alpha \right] = \frac{2}{\pi \theta} \left[\cos \alpha \theta \Big|_0^1 \right] = \frac{2(1 - \cos \theta)}{\pi \theta^2} \quad (ii)$$

$$\text{Now } F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha \theta d\theta = \int_0^\infty \frac{2(1 - \cos \theta)}{\pi \theta^2} \cos \alpha \theta d\theta$$

$\therefore$ From (i) and (ii), we have

$$\frac{2}{\pi} \int_0^\infty \frac{1 - \cos \theta}{\theta^2} \cos \alpha \theta d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Now lettering $\alpha \rightarrow 0$, we get $\frac{2}{\pi} \int_0^\infty \frac{1 - \cos \theta}{\theta^2} d\theta = 1$

$$\int_0^\infty \frac{2 \sin^2 \theta / 2}{\theta^2} d\theta = \pi / 2 \quad [\text{Put } \theta / 2 = t, \text{ so that } d\theta = 2dt]$$

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \pi / 2$$

EXERCISE 5.1 :

1.

Using the Fourier integral representation show that

$$(i) \int_0^\infty \frac{\omega \sin x\omega}{1 + \omega^2} d\omega = \frac{\pi}{2} e^{-x} \quad (x > 0)$$

$$(ii) \int_0^\infty \frac{\cos \omega x}{1 + \omega^2} d\omega = \frac{\pi}{2} e^{-x} \quad (x \geq 0)$$

2. Prove that $\int_0^\infty \frac{\sin \pi \alpha \sin \alpha \theta}{1-\alpha^2} d\alpha = \begin{cases} \frac{1}{2} \pi \sin \theta, & 0 \leq \theta \leq \pi \\ 0, & \theta > \pi \end{cases}$
3. Find the Fourier transform of $e^{-x^2/2}$, $-\infty < x < \infty$
4. Find the Fourier transforms of $f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$
5. Given $F(e^{-x^2}) = \sqrt{\pi} e^{-s^2/4}$, find the Fourier transform of
 (i) $e^{-x^2/3}$ (ii) $e^{-4(x-3)^2}$
6. Find the Fourier cosine transforms of $f(x) = e^{-x^2/2}$
7. Find the Fourier transforms of $f(x) = \begin{cases} 1, & 0 \leq x < a \\ 0, & x \geq a \end{cases}$
8. Find the Fourier sine transform of $\frac{1}{x(x^2 + a^2)}$
9. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x dx = e^{-\alpha}$

ANSWERS 5.1

3. $\sqrt{2\pi} e^{-\frac{s^2}{2}}$
4. $\frac{4}{s^3} (\sin as - sa \cos sa)$
- 5 (i) $\sqrt{3\pi} e^{-\frac{3s^2}{4}}$ (ii) $\frac{\sqrt{\pi}}{2} e^{\left(\frac{3is-s^2}{4}\right)}$
6. $\frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{4}}$
7. $\frac{1-\cos as}{s}, \frac{\sin as}{s}$
8. $\frac{\pi}{2a^2} (1 - e^{-as})$
9. $\frac{2}{\pi(1+x^2)}$

5.6(1) CONVOLUTION

The convolution of two function $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du = h(x)$$

- (2) **Convolution theorem for Fourier transform.** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms i.e.

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$$

we have $F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u) g(x-u) du\right\}$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(u) g(x-u) du \right\} e^{isx} dx = \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} g(x-u) e^{isx} dx \right\} du$$

[Changing the order of integration]

$$= \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} e^{is(x-u)} \cdot g(x-u) d(x-u) \right\} e^{isu} du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u) \left\{ \int_{-\infty}^{\infty} e^{ist} g(t) dt \right\} du \text{ where } x-u=t$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u) du \cdot F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x) dx \cdot F\{g(x)\} = F\{f(x)\} \cdot F\{g(x)\}$$

5.7

PARSEVAL'S THEOREM FOR FOURIER TRANSFORMS

If the fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

(i) $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \bar{G}(s) ds = \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx$

(ii) $\frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

Where bar implies the complex conjugate,

(i) $\int_{-\infty}^{\infty} f(x) \bar{g}(x) dx = \int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) e^{isx} ds \right\} dx$

[Changing the inversion formula of Fourier transform]
 [Changing the order of integration]

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) \left\{ \int_{-\infty}^{\infty} f(x) e^{isx} dx \right\} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G}(s) F(s) ds, \text{ by definition of Fourier transform.}$$

(ii) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \overline{F}(s) ds = \int_{-\infty}^{\infty} f(x) \overline{f}(x) dx \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Example.14 Using Parseval's Theorem, prove that

$$(i) \int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)} \quad (ii) \int_0^{\infty} \frac{t^2}{(t^2+1)^2} dt = \frac{\pi}{4}$$

$$(iii) \int_0^{\infty} \frac{\sin at}{t(a^2+t^2)} dt = \frac{\pi}{2} \cdot \frac{1-e^{-a^2}}{a^2}$$

Soln. (i) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$, Then $F_c = \frac{a}{a^2+s^2}$, $G_c(s) = \frac{b}{b^2+s^2}$
Now using Parseval's theorem for Fourier cosine transforms, i.e.

$$\frac{2}{\pi} \int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx$$

We have $\frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$

or $\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left| \frac{e^{-(a+b)x}}{-(a+b)} \right|_0^{\infty} = \frac{1}{a+b}$

Thu $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$

(ii) Let $f(x) = \frac{x}{x^2+1}$, so that $F_s\{f(x)\} = \frac{\pi}{2} e^{-s}$

Now using Parseval's theorem for sine transform, i.e.

$$\frac{2}{\pi} \int_0^{\infty} [F_s\{f(x)\}]^2 ds = \int_0^{\infty} \{f(x)\}^2 dx \quad \text{or} \quad \int_0^{\infty} \left(\frac{x}{x^2+1} \right)^2 dx = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\pi}{2} e^{-s} \right)^2 ds$$

$$= \frac{\pi}{2} \left[e^{-2s} / -2 \right]_0^{\infty} = \frac{\pi}{-4} (0-1) = \frac{\pi}{4}$$

Hence $\int_0^{\infty} \frac{t^2}{(t^2+1)^2} dt = \frac{\pi}{4}$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$ then $F_c(s) = \frac{a}{a^2+s^2}$, $G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi} \int_0^{\infty} \frac{a \sin as}{s(a^2+s^2)} ds = \int_0^a e^{-ax} \cdot 1 dx = \frac{1-e^{-a^2}}{a}$

Thus $\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)} dt = \frac{\pi}{2a^2} (1-e^{-a^2})$

EXERCISE 5.2 :

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$
2. If $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and $F(s) = \frac{2 \sin sa}{s}$, ($s \neq 0$) then prove that $\int_0^{\infty} \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}$
3. Using Parseval's theorem, show that
 - (i) $\int_0^{\infty} \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$
 - (ii) $\int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$
4. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ and hence show that $\int_0^{\infty} \frac{\sinh^4 t}{t^4} dt = \frac{\pi}{3}$

ANSWERS 5.2

4 (i.) $\frac{\pi}{2}$ (ii) $\frac{\pi}{2}$

5.8

RELATIONSHIP BETWEEN FOURIER AND LAPLACE TRANSFORMS

If

$$f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}$$

and $F\{f(t)\} = L\{g(t)\}$

we have

$$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t) dt = \int_{-\infty}^0 e^{ist} \cdot 0 \cdot dt + \int_0^{\infty} e^{ist} \cdot e^{-xt} \cdot g(t) dt$$

$$= \int_0^{\infty} e^{(is-x)t} g(t) dt = \int_0^{\infty} e^{-pt} g(t) dt = L\{g(t)\}$$

where $p = x - is$

Hence the Fourier transform of $f(t)$ [Defined by (i)] is the Laplace transform of $g(t)$.

5.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x, t)] = \int_{-\infty}^{\infty} u e^{isx} dx$$

Then the Fourier transform of $\partial^2 u / \partial x^2$, i.e.

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \left[e^{isx} \frac{\partial u}{\partial x} - is e^{isx} u \right]_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} u e^{isx} dx,$$

on applying the general rule of integration by parts. If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $+\infty$, then

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F[u] \quad (1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s(u)_{x=0} - s^2 F_s[u] \quad (2)$$

$$\text{and } F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c[u] \quad (3)$$

In general, the Fourier transform of the n th derivative of

$$F\left[\frac{d^n f}{dx^n}\right] = (-is)^n F[f(x)]$$

$f(x)$ is given by

(4)

Provided the first $n-1$ derivatives vanish as $x \rightarrow \pm\infty$

For $F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x) e^{isx} dx$

$$= \left[e^{isx} f^{n-1} - is e^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \dots \right]_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f e^{isx} dx$$

by the general rule of integration by parts, whence follows.

5.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transforms of $f(x)$ be $\bar{f}(s)$ so that $\bar{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\begin{aligned} \int_{a-ir}^{a+ir} e^{xs} \bar{f}(s) ds &= \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st} dt ds \\ &= \int_0^{\infty} f(t) e^{-(a-iu)t} dt \int_{-r}^r e^{ax} e^{-ixu} e^{-it} e^{it} du \\ &= \int_0^{\infty} f(t) e^{-(a-iu)t} dt \int_{-r}^r e^{ax} e^{-ixu} e^{-it} e^{it} du \end{aligned}$$

$$= ie^{ax} \int_0^{\infty} f(t) e^{-it} dt \int_{-r}^r e^{-ixu} e^{iut} du, \quad \text{where } \phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Proceeding to limits as $r \rightarrow \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds = ie^{ax} \cdot 2\pi \phi(x) = 2\pi ie^{ax} \cdot e^{-ax} f(x) \quad \text{for } x > 0$$

Hence $f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds \quad (x > 0)$

Which is called the complex inversion formula. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel imaginary axis in the complex plane $s = x + iy$ such that all singularities of $\bar{f}(s)$ are to the left of LM. Let us take a contour C which is composed of the line LM and the semi-circle i.e. MNL. Then from (2)

$$\frac{1}{2\pi} \int_{LM} e^{xs} \bar{f}(s) ds = \frac{1}{2\pi i} \int_C e^{xs} \bar{f}(s) ds - \frac{1}{2\pi i} \int_{MNL} e^{xs} \bar{f}(s) ds$$

The integral over C' tends to zero as $r \rightarrow \infty$ (under certain). Therefore

$$f(x) = \lim_{t \rightarrow \infty} \frac{1}{2\pi i} \int_c e^{xs} \bar{f}(s) ds$$

= sum of the residues of $e^{xs} \bar{f}(s)$ at the poles of $f(s)$

Example 15 Evaluate $L^{-1} \left\{ \frac{1}{(s-1)(s^2+1)} \right\}$ by method of residues.

Soln. Since $\left| \frac{1}{(s-1)(s^2+1)} \right| \sim \left| \frac{1}{s^3} \right|$ for $|s| \rightarrow \infty$, therefore

$$L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \text{sum of Res} \left[\frac{e^{xs}}{(s-1)(s^2+1)} \right] \text{ at the poles } s = 1, \pm i$$

$$\text{Now } (Res)_{s=1} = \lim_{s \rightarrow 1} \left[\frac{(s-1) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^x}{2}$$

$$(Res)_{s=i} = \lim_{s \rightarrow i} \left[\frac{(s-1) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^{ix}}{(i-1)(-i-1)} = -\frac{1}{2} \cdot \frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get

$$(Res)_{s=-i} = -\frac{1}{2} \cdot \frac{e^{-ix}}{1-i}$$

$$L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \frac{e^x}{2} - \frac{1}{2} \left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i} \right) = \frac{1}{2} (e^x - \sin x - \cos x)$$

EXERCISE 5.3 :

Using the method of residues, evaluate the inverse Laplace transform

1. $\frac{1}{s^2(s^2-a^2)}$

2. $\frac{1}{(s-1)^2(s^2+1)}$

3. $\frac{1}{(s^2+1)^2}$

4. $\frac{1}{(s+1)(s-2)^2}$

ANSWERS 5.3

1. $(\sinh at - at)/a^2$ 2. $1/e^x (x-1) + \cos x$ 3. $(\sin t - t \cos t)/2$ 4. $\frac{1}{9}(e^{-t} + e^{2t}(3t-1))$

MULTIPLE CHOICE QUESTIONS

1. Fourier cosine transform $f(t)$ is

(a) $\int_0^\infty f(t) \cos st dt$

(b) $\int_0^\infty f(t) \cos t dt$

(c) $\int_0^\infty f(t) \sin st dt$

(d) $\int_0^\infty f(t) dt$

2. Fourier cosine transform of $1/x$ is,...

(a) $\frac{s}{2}$

(b) $\frac{s^2}{2}$

(c) $\frac{s^4}{2}$

(d) $\frac{s^3}{2}$

3. If Fourier cosine transform of $f(x)$ is $F(s)$, then the inversion formula is,...

(a) $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{xs} ds$

(b) $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{-is} ds$

(c) $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{-isx} ds$

(d) $f(x) = \frac{1}{\pi} \int_{-\infty}^\infty F(s) e^{-isx} ds$

4. $F[x^n f(x)] = \dots\dots\dots$

(a) $\int_{-\infty}^\infty t^n f(t) dt$

(b) $\int_{-\infty}^\infty t f(t) e^{ist} dt$

(c) $\int_{-\infty}^\infty f(t) e^{ist} dt$

(d) $\int_{-\infty}^\infty t^n f(t) e^{ist} dt$

5. If $F[f(x)] = F'(s)$, then $F\{f(x-a)\} = \dots\dots\dots$

(a) $e^{isa} F(s)$

(b) $e^a F(s)$

(c) $eF(s)$

(d) e^{isa}

6. Fourier sine integral representation of a function $f(x)$ is given by,...
- (a) $\frac{2}{\pi} \int_{-\infty}^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t d\lambda$ (b) $\frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda$
- (c) $\int_{-\infty}^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda$ (d) $\frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda$
7. If $F_c[f(ax)] = k F_c(s/a)$ then $K = \dots$
- (a) $\frac{-1}{a}$ (b) $\frac{1}{2a}$
- (c) $\frac{1}{a}$ (d) a
7. If $F_c[f(ax)] = k F_c(s/a)$ then $K = \dots$
- (a) $\frac{-1}{a}$ (b) $\frac{1}{2a}$
- (c) $\frac{1}{a}$ (d) a
8. Fourier transforms of second derivative of $u(x, t)$ is,.....
- (a) $s^2(F(u))$ (b) $-s^3(F(u))$
- (c) $s(F(u))$ (d) $-s^2(F(u))$
9. If $f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is
- (a) $\frac{2}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\pi} \sin(\lambda t) dt$ (b) $\frac{e}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\pi} \sin(\lambda t) dt$
- (c) $\frac{1}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\pi} \sin(\lambda t) dt$ (d) $\frac{3}{\pi} \int_0^{\infty} \sin(\lambda x) d\lambda \int_0^{\pi} \sin(\lambda t) dt$

10. If Fourier transform of $f'(x)$ in the interval $(0, l)$ is,....
- (a) $\frac{n\pi}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$ (b) $-\frac{n\pi}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$
- (c) $\int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$ (d) $-\frac{n}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$
11. If $F(\lambda)$ is the Fourier transform of $f(x)$ then the Fourier transform of $f(ax)$ is
- (a) $F\left(\frac{\lambda}{a}\right)$ (b) $\frac{1}{a} F\left(\frac{1}{a}\right)$
- (c) $\frac{1}{a} F\left(\frac{\lambda}{a}\right)$ (d) $\frac{1}{2a} F\left(\frac{\lambda}{a}\right)$
12. $\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds$
- (a) True (b) none
- (c) 0 (d) false
13. Fourier transform is a linear operation.
- (a) True (b) none
- (c) 0 (d) false
14. $F_s[xf(x)] = -\frac{d}{dx} F_c(s)$
- (a) false (b) True
- (c) 0 (d) none
15. If $f(x)$ has the Fourier transform $f(s)$ then $f(x) \cos ax$ has Fourier transform is
- (a) $f(s-a) + f(s+a)$ (b) $\frac{1}{2} f(s+a) + \frac{1}{2} f(a-s)$
- (c) $\frac{1}{2} f(s-a) + \frac{1}{2} f(s+a)$ (d) None

16. The fourier transform of $f^{-1}(x)$, the derivative of $F(x)$, and $f(s)$ is fourier transform $f(x)$ then derivative of $f(x)$ is,....
- (a) $sf(s)$ (b) $f(s)$
(c) $-isf(s)$ (d) $isf(s)$
17. If $f(s)$ is the fourier transform of $f(x)$ then $\frac{1}{a}f\left(\frac{s}{a}\right)$ is the fourier transform of
- (a) $F(ax)$ (b) $F(ax^2)$
(c) $F(s)$ (d) $F(a^2x)$
18. The sine transforms of $2e^{-5x} + 5e^{-2x}$ is.
- (a) $s\left[\frac{2}{s^2+25} + \frac{10}{s^2+4}\right]$ (b) $s\left[\frac{2}{s^2+25} + \frac{5}{s^2+4}\right]$
(c) $10\left[\frac{1}{s^2+4} + \frac{1}{s^2+25}\right]$ (d) None
19. The finite sine transform of e^{ax} is
- (a) $\frac{s}{a^2+s^2}\left[1+(-1)^s e^{a\pi}\right]$ (b) $\frac{1}{a^2+s^2}\left[1+(1)^s e^{a\pi}\right]$
(c) $\frac{s}{a^2+s^2}\left[1-(-1)^s e^{a\pi}\right]$ (d) $\frac{1}{a^2+s^2}\left[1-(-1)^s e^{a\pi}\right]$
20. The finite sine transform of $[\sin ax]$ if $a \neq s$ and a, s are integers is,...
- (a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) 1 (d) 0
21. The finite cosine transform of $\frac{\pi}{3} - x + \frac{x^2}{2\pi}$ is,...
- (a) $\frac{1}{s^2}$ (b) $\frac{1}{s}$

- (c) $\frac{1}{s^3}$ (d) s^2
22. The finite sine transform of $\left(\frac{x}{\pi}\right)$ is
- (a) $(-1)^{s+2} \frac{1}{s}$ (b) $(-1)^{s+1} \frac{1}{s}$
(c) $(-1)^{s+1} \frac{1}{s^2}$ (d) None
23. If $F[f(x)] = F(\lambda)$ then $F[f(ax)] =$ where $a \neq 0$
- (a) $\frac{1}{|a|}F[a]$ (b) $\frac{1}{|a|}F[\lambda]$
(c) $\frac{1}{|a|}F\left[\frac{\lambda}{a}\right]$ (d) None
24. If $F[f(x)] = F(\lambda)$, then $F[f(x-a)] =$
- (a) $e^{i\lambda a}F(\lambda)$ (b) $e^{\lambda a}F(\lambda)$
(c) $e^{-i\lambda}F(\lambda)$ (d) $e^{-i\lambda a}F(\lambda)$
25. If $f_s(n) = \frac{K(-1)^{n-1}}{n^2}$, $n = 1, 2, \dots, K$ is integer $0 < x < l$ then $f(x) =$
- (a) $\frac{2k}{l} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi x}{l}$ (b) $\frac{2}{l} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi x}{l}$
(c) $\frac{2k}{l} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left(\frac{n\pi}{2}\right)$ (d) None
26. If $F[f(ax)] = \frac{1}{|a|}F\left(\frac{\lambda}{a}\right)$, $a \neq 0$ then $F[e^{-|x|}] =$

(a) $\frac{1}{1+\lambda^2}$

(b) $\frac{2}{1+\lambda^2}$

(c) $\frac{4}{1^2+\lambda^2}$

(d) $\frac{3}{1+\lambda^2}$

27. If $F[f(ax)] = \frac{1}{|a|} F\left(\frac{\lambda}{a}\right)$, $a \neq 0$ then $F[e^{-|2x|}] =$

(a) $\frac{2}{4+\lambda^2}$

(b) $\frac{4}{2+\lambda^2}$

(c) $\frac{4}{4+\lambda^2}$

(d) $\frac{2}{3^2+\lambda^2}$

ANSWER KEY (MULTIPLE CHOICE QUESTIONS):

01.(a)	02.(b)	03.(c)	04.(d)	05.(a)	06.(b)
07.(c)	08.(d)	09.(a)	10.(b)	11.(c)	12.(d)
13.(a)	14.(b)	15.(c)	16.(d)	17.(a)	18.(b)
19.(c)	20.(d)	21.(a)	22.(b)	23.(c)	24.(d)
25.(a)	26.(b)	27.(c)			

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