

BAHIRJI SMARAK MAHAVIDYALAYA

BASMATHNAGAR

Dist. Hingoli, Maharashtra

B.Sc. MATHEMATICS

First Year | Semester – I

DSC-1 : TOPICS IN ALGEBRA

Course Code: SMATCT1101

STUDY NOTES

Theory • Solved Examples • Practice Questions

Item	Details
Programme	Certificate Course in Mathematics (B.A./B.Sc. I Year, First Level 4.5)
Course Type	DSC (Discipline Specific Core)
Credits / Marks	2 Credits Max. Marks 50 Min. Passing Marks 20
L-T-P	2 – 0 – 0 (hours per week)
Prerequisite	Elementary knowledge of Algebra
University	S.R.T.M. University, Nanded (Faculty of Science & Technology)

Prepared for the students of B.Sc. First Year

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BY

Dr Narsingh

About This Course

Course Outcomes (COs)

After successfully completing this course, a student will be able to:

CO	Outcome	Covered In
CO1	Sort one-one and onto functions and compute equivalence relations.	Modules I & II
CO2	Compute the composite of functions.	Module II
CO3	Compute REF, RREF and rank of any ordered matrix.	Module III
CO4	Solve a linear system of equations and apply the Cayley–Hamilton theorem.	Module IV

Syllabus at a Glance

Module	Topic	Lectures
I	Sets, Functions and Relations: basic terminology, operations on sets, Cartesian product, relations, types of relations, equivalence relations	08
II	Functions: one-one, onto, bijections, composition, inverse, equivalence classes, partitions of a set	07
III	Elementary Operations and Rank of a Matrix: elementary operations, equivalent matrices, row rank and column rank, rank, echelon form, row reduced echelon form	07
IV	System of Linear Equations and Cayley–Hamilton Theorem: homogeneous and non-homogeneous systems, characteristic values and vectors, Cayley–Hamilton theorem	08

Prescribed Books

- **Text Book 1 (Modules I and II):** Ajit Kumar, S. Kumaresan and Bhaba Kumar Sarma, *A Foundation Course in Mathematics*, Narosa Publishing House.
- **Text Book 2 (Modules III and IV):** Om P. Chug, Kulbhushan Prakash, A. D. Gupta, *Topics in Algebra*, Anmol Publications, New Delhi, First Edition 1997.
- **Reference:** Shanti Narayan, *A Text Book of Matrices*, S. Chand & Company.
- **Reference:** Seymour Lipschutz, *Theory and Problems of Linear Algebra*, 3rd ed., Tata McGraw-Hill.
- **Reference:** K. B. Datta, *Matrix and Linear Algebra*, PHI Learning.

How to Use These Notes

1. Read the definitions and theorems first; they are placed in shaded boxes for quick revision.
2. Study each solved example, then cover the solution and try it yourself.
3. Attempt the practice questions at the end of every module. Answers are given for numerical problems.
4. For matrix topics, practise row operations daily; speed and accuracy come only through practice.

Remember

These notes follow the university syllabus topic by topic. Use them together with the prescribed textbooks and your class lectures.

Exam Strategy (50 Marks)

- Learn definitions and theorem statements word for word; they carry easy marks.
- Write the row operation (such as $R_2 \rightarrow R_2 - 2R_1$) beside every step in matrix problems.

MODULE I: Sets, Relations and Equivalence Relations

Lectures: 08 | Text Book 1: Chapter 2 (Articles 2.1, 2.2, 2.5) and Chapter 4 (Articles 4.1–4.3)

1.1 Basic Terminology of Sets

Set theory is the foundation on which almost all of modern mathematics is built. A set is understood as a well-defined collection of distinct objects. “Well-defined” means that for any given object we can decide definitely whether it belongs to the collection or not.

Definition: Set

A **set** is a well-defined collection of distinct objects, called **elements** or **members** of the set. Sets are denoted by capital letters $A, B, C, \dots$ and elements by small letters $a, b, c, \dots$. If x is an element of A we write $x \in A$; otherwise $x \notin A$.

Standard Number Sets

Symbol	Set	Description
$\mathbb{N}$	Natural numbers	$\{1, 2, 3, \dots\}$
$\mathbb{Z}$	Integers	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{Q}$	Rational numbers	$\{p/q : p, q \in \mathbb{Z}, q \neq 0\}$
$\mathbb{R}$	Real numbers	All rational and irrational numbers
$\mathbb{C}$	Complex numbers	$\{a + ib : a, b \in \mathbb{R}\}$

Ways of Describing a Set

- **Roster (tabular) form:** list the elements, e.g. $A = \{1, 2, 3, 4, 5\}$.
- **Set-builder form:** state a property, e.g. $A = \{x \in \mathbb{N} : x \leq 5\}$, read as “the set of all x in $\mathbb{N}$ such that $x \leq 5$ ”.

Types of Sets

- **Empty (null) set $\emptyset$:** contains no element. Example: $\{x \in \mathbb{R} : x^2 = -1\}$.
- **Singleton set:** contains exactly one element, e.g. $\{7\}$.
- **Finite set:** has a finite number of elements; $n(A)$ or $|A|$ denotes the number of elements.
- **Infinite set:** not finite, e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$.
- **Equal sets:** $A = B$ if they have exactly the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.
- **Universal set U :** the set containing all objects under discussion in a particular problem.

Definition: Subset

A is a **subset** of B , written $A \subseteq B$, if every element of A is also an element of B , i.e. $x \in A \Rightarrow x \in B$. If $A \subseteq B$ and $A \neq B$, then A is a **proper subset**, written $A \subset B$.

Definition: Power Set

The **power set** of A , denoted $P(A)$, is the set of all subsets of A . If A has n elements then $P(A)$ has 2^n elements.

Example 1.1

Let $A = \{a, b, c\}$. Then $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Here $n(A) = 3$ and $n(P(A)) = 2^3 = 8$.

1.2 Operations on Sets

Let A and B be subsets of a universal set U . The following operations are defined.

Operation	Symbol	Definition
Union	$A \cup B$	$\{x : x \in A \text{ or } x \in B\}$
Intersection	$A \cap B$	$\{x : x \in A \text{ and } x \in B\}$
Difference	$A - B$ (or $A \setminus B$)	$\{x : x \in A \text{ and } x \notin B\}$
Complement	A' (or A^c)	$U - A = \{x \in U : x \notin A\}$
Symmetric difference	$A \Delta B$	$(A - B) \cup (B - A)$

Two sets A and B are called **disjoint** if $A \cap B = \emptyset$.

Algebra of Sets (Laws)

Law	Statement
Idempotent	$A \cup A = A, A \cap A = A$
Commutative	$A \cup B = B \cup A, A \cap B = B \cap A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A, A \cap U = A$
Complement	$A \cup A' = U, A \cap A' = \emptyset, (A')' = A$
De Morgan	$(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$

Theorem: De Morgan's Law

For any sets A and B , $(A \cup B)' = A' \cap B'$.

Proof. Let $x \in (A \cup B)'$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Hence $x \in A'$ and $x \in B'$, i.e. $x \in A' \cap B'$. Therefore $(A \cup B)' \subseteq A' \cap B'$.

Conversely, let $x \in A' \cap B'$. Then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$, i.e. $x \in (A \cup B)'$. Therefore $A' \cap B' \subseteq (A \cup B)'$. By mutual inclusion, $(A \cup B)' = A' \cap B'$. ■

Theorem: Distributive Law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. $x \in A \cap (B \cup C) \Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C).$ ■

Theorem: Cardinality of a Union

For finite sets, $n(A \cup B) = n(A) + n(B) - n(A \cap B).$

For three sets, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$

Example 1.2

Let $U = \{1, 2, \dots, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$. Find $A \cup B$, $A \cap B$, $A - B$, $B - A$, A' and $A \Delta B$.

Solution. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$; $A \cap B = \{4, 5\}$; $A - B = \{1, 2, 3\}$; $B - A = \{6, 7\}$; $A' = \{6, 7, 8, 9, 10\}$; $A \Delta B = \{1, 2, 3\} \cup \{6, 7\} = \{1, 2, 3, 6, 7\}.$

Example 1.3

In a class of 60 students, 35 like Mathematics, 30 like Physics and 10 like both. How many like neither?

Solution. $n(M \cup P) = 35 + 30 - 10 = 55$. Students liking neither = $60 - 55 = 5$.

1.3 Cartesian Product of Sets

Definition: Ordered Pair and Cartesian Product

An **ordered pair** (a, b) is a pair in which the order matters: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

The **Cartesian product** of sets A and B is $A \times B = \{(a, b) : a \in A, b \in B\}.$

Example 1.4

Let $A = \{1, 2\}$ and $B = \{x, y, z\}$. Then $A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$ and $B \times A = \{(x, 1), (y, 1), (z, 1), (x, 2), (y, 2), (z, 2)\}$. Clearly $A \times B \neq B \times A$, although $n(A \times B) = n(B \times A) = 6$.

Properties of Cartesian Product

- If $n(A) = m$ and $n(B) = n$, then $n(A \times B) = mn$.
- $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.
- In general $A \times B \neq B \times A$; equality holds only if $A = B$ (or one set is empty).
- $A \times (B \cup C) = (A \times B) \cup (A \times C).$
- $A \times (B \cap C) = (A \times B) \cap (A \times C).$
- If $A \subseteq B$ then $A \times C \subseteq B \times C$ for every set C .

- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ represents the Cartesian plane; the Cartesian product of n sets gives ordered n -tuples.

Proof of $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $(a, b) \in A \times (B \cap C) \Leftrightarrow a \in A$ and $b \in B \cap C \Leftrightarrow a \in A, b \in B$ and $b \in C \Leftrightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C \Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$. ■

1.4 Relations on Sets

Definition: Relation

A **relation** R from a set A to a set B is a subset of $A \times B$. If $(a, b) \in R$ we write $a R b$ ("a is related to b"). A relation from A to A is called a relation **on** A .

Definition: Domain, Range and Inverse

Domain of $R = \{a \in A : (a, b) \in R \text{ for some } b\}$. **Range** of $R = \{b \in B : (a, b) \in R \text{ for some } a\}$.

Inverse relation $R^{-1} = \{(b, a) : (a, b) \in R\}$, a relation from B to A .

Example 1.5

Let $A = \{1, 2, 3, 4\}$ and define R on A by $a R b \Leftrightarrow a < b$. Then $R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$. $\text{Domain}(R) = \{1, 2, 3\}$, $\text{Range}(R) = \{2, 3, 4\}$, and $R^{-1} = \{(2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$.

Remember

If $n(A) = m$ and $n(B) = n$, the number of different relations from A to B is 2^{mn} , because each of the mn ordered pairs of $A \times B$ may either be included in R or left out.

1.5 Types of Relations

Let R be a relation on a set A . We examine the following properties.

Type	Definition	Meaning
Reflexive	$a R a$ for all $a \in A$	Every element is related to itself
Symmetric	$a R b \Rightarrow b R a$	Relation works both ways
Anti-symmetric	$a R b$ and $b R a \Rightarrow a = b$	Two-way relation only between equal elements
Transitive	$a R b$ and $b R c \Rightarrow a R c$	Relation can be chained

Special Relations

- **Identity (diagonal) relation:** $I_A = \{(a, a) : a \in A\}$.
- **Universal relation:** $R = A \times A$ (every element related to every element).
- **Empty (void) relation:** $R = \emptyset$ (no element is related to any element).

Example 1.6

Examine the relations on $\mathbb{Z}$: (i) $a R b \Leftrightarrow a \leq b$; (ii) $a R b \Leftrightarrow a = b$; (iii) $a R b \Leftrightarrow |a - b| \leq 1$.

Solution. (i) Reflexive ($a \leq a$), anti-symmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$), transitive; not symmetric ($1 \leq 2$ but $2 \not\leq 1$).

(ii) Reflexive, symmetric, anti-symmetric and transitive.

(iii) Reflexive and symmetric. Not transitive: $|1 - 2| \leq 1$ and $|2 - 3| \leq 1$ but $|1 - 3| = 2 > 1$.

Example 1.7

Let $A = \{1, 2, 3\}$ and $R = \{(1,1), (2,2), (1,2), (2,1)\}$. Is R reflexive? symmetric? transitive?

Solution. Not reflexive, because $(3,3) \notin R$. Symmetric: $(1,2) \in R$ and $(2,1) \in R$. Transitive: checking all chains, e.g. $(1,2), (2,1) \Rightarrow (1,1) \in R$, and so on. So R is symmetric and transitive but not reflexive.

1.6 Equivalence Relations

Definition: Equivalence Relation

A relation R on a non-empty set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive**.

Equivalence relations capture the idea of “sameness” in some respect: being the same age, having the same remainder on division by n , living in the same city, etc.

Standard Examples

- Equality ($=$) on any set.
- $a \sim b \Leftrightarrow a$ and b have the same remainder on division by 5 (on $\mathbb{Z}$).
- Similarity of triangles, parallelism of lines (a line is taken parallel to itself).
- Congruence of integers modulo n .

Non-Examples

- “Less than or equal to” on $\mathbb{R}$ (not symmetric).
- “Perpendicular to” on lines (not reflexive, not transitive).
- “Is a friend of” (generally not transitive).

Definition: Congruence Modulo n

For a fixed positive integer n and $a, b \in \mathbb{Z}$, we say a is congruent to b modulo n , written $a \equiv b \pmod{n}$, if n divides $(a - b)$.

Theorem

Congruence modulo n is an equivalence relation on $\mathbb{Z}$.

Proof. Let $a, b, c \in \mathbb{Z}$.

1. **Reflexive:** $a - a = 0 = n \cdot 0$, so $n \mid (a - a)$. Hence $a \equiv a \pmod{n}$.
2. **Symmetric:** Let $a \equiv b \pmod{n}$. Then $a - b = nk$ for some $k \in \mathbb{Z}$, so $b - a = n(-k)$. Hence $b \equiv a \pmod{n}$.
3. **Transitive:** Let $a \equiv b$ and $b \equiv c \pmod{n}$. Then $a - b = nk$ and $b - c = nm$. Adding, $a - c = n(k + m)$. Hence $a \equiv c \pmod{n}$.

Since all three properties hold, $\equiv \pmod{n}$ is an equivalence relation. ■

Example 1.8

Show that the relation R on the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation.

Solution. *Reflexive:* $|a - a| = 0$ is even. *Symmetric:* $|a - b| = |b - a|$, so if one is even the other is. *Transitive:* if $|a - b|$ and $|b - c|$ are even then $a - c = (a - b) + (b - c)$ is a sum of two even numbers, hence even. So R is an equivalence relation. It relates numbers of the same parity: $\{1, 3, 5\}$ and $\{2, 4\}$.

Example 1.9

On the set $\mathbb{N} \times \mathbb{N}$ define $(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$. Prove that $\sim$ is an equivalence relation.

Solution. *Reflexive:* $a + b = b + a$, so $(a, b) \sim (a, b)$. *Symmetric:* if $a + d = b + c$ then $c + b = d + a$, so $(c, d) \sim (a, b)$. *Transitive:* let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Adding: $a + d + c + f = b + c + d + e$, so $a + f = b + e$, i.e. $(a, b) \sim (e, f)$. Hence $\sim$ is an equivalence relation. (This is how integers are constructed from natural numbers.)

Example 1.10

Let A be the set of all lines in a plane and let $L_1 R L_2$ if L_1 is parallel to L_2 or $L_1 = L_2$. Show R is an equivalence relation.

Solution. Reflexive since $L = L$. Symmetric since “parallel” is symmetric. Transitive: if $L_1 \parallel L_2$ and $L_2 \parallel L_3$ then $L_1 \parallel L_3$ (or they coincide). Hence R is an equivalence relation.

1.7 Summary of Module I

- A relation from A to B is any subset of $A \times B$; there are 2^{mn} of them.
- Four key properties: reflexive, symmetric, anti-symmetric, transitive.
- An equivalence relation is reflexive + symmetric + transitive.
- To prove a relation is an equivalence relation, check each of the three properties separately; to disprove, give one counter-example for any property.

1.8 Practice Questions: Module I

1. Define a power set. Find the power set of $\{1, 2, 3\}$ and write the number of its subsets.
2. Prove De Morgan's law $(A \cap B)' = A' \cup B'$.
3. In a survey of 100 people, 65 read newspaper X , 55 read newspaper Y and 30 read both. How many read neither? (Ans: 10)

4. If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, write $A \times B$ and $B \times A$. Is $A \times B = B \times A$?
5. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
6. Define reflexive, symmetric and transitive relations with one example each.
7. Let $R = \{(a, b) : a, b \in \mathbb{Z}, a - b \text{ is divisible by } 3\}$. Show that R is an equivalence relation.
8. Check whether the relation “ x is a divisor of y ” on $\mathbb{N}$ is reflexive, symmetric, anti-symmetric or transitive.
9. Give an example of a relation that is symmetric and transitive but not reflexive.
10. If $n(A) = 3$ and $n(B) = 2$, how many relations from A to B exist? (Ans: 64)

MODULE II: Functions

Lectures: 07 | Text Book 1: Chapter 3 (Articles 3.1–3.4) and Chapter 4 (Article 4.4)

2.1 Functions: Basic Ideas

Definition: Function

Let A and B be non-empty sets. A **function** f from A to B , written $f : A \rightarrow B$, is a rule that assigns to each element $x \in A$ exactly one element $f(x) \in B$.

A is the **domain**, B is the **codomain**, and the set $f(A) = \{f(x) : x \in A\} \subseteq B$ is the **range** (image) of f .

In terms of relations, a function $f : A \rightarrow B$ is a relation from A to B such that (i) every $a \in A$ is related to some $b \in B$, and (ii) no $a \in A$ is related to two different elements of B .

Example 2.1

- (i) $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is a function; its range is $[0, \infty)$.
- (ii) The rule " $x \mapsto \pm\sqrt{x}$ " is *not* a function, since one input gives two outputs.
- (iii) The identity function $I_A : A \rightarrow A$, $I_A(x) = x$.
- (iv) A constant function $f(x) = c$ for all x .

Two functions $f, g : A \rightarrow B$ are **equal** if $f(x) = g(x)$ for all $x \in A$ (same domain, same codomain, same rule).

2.2 One-One, Onto Functions and Bijections

Definition: One-one (Injective) Function

$f : A \rightarrow B$ is **one-one** if different elements of A have different images:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ for all } x_1, x_2 \in A.$$

Definition: Onto (Surjective) Function

$f : A \rightarrow B$ is **onto** if every element of B is the image of at least one element of A , i.e. $f(A) = B$: for every $y \in B$ there exists $x \in A$ with $f(x) = y$.

Definition: Bijection

A function that is both one-one and onto is called a **bijective function** (one-to-one correspondence).

How to Test

- **One-one:** assume $f(x_1) = f(x_2)$ and show $x_1 = x_2$. To disprove, find two different inputs with the same output.
- **Onto:** take an arbitrary y in the codomain, solve $y = f(x)$ for x , and check that x lies in the domain. To disprove, find a y with no pre-image.

Example 2.2

Examine $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3$ for one-one and onto.

Solution. *One-one:* $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$. *Onto:* for $y \in \mathbb{R}$, solve $y = 2x + 3$ to get $x = (y - 3)/2 \in \mathbb{R}$, and $f(x) = y$. Hence f is a **bijection**.

Example 2.3

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$. Is f one-one? onto?

Solution. $f(2) = f(-2) = 4$ but $2 \neq -2$, so f is *not one-one*. Also -1 has no real square root, so -1 is not in the range; f is *not onto*.

If we restrict the function to $g : [0, \infty) \rightarrow [0, \infty), g(x) = x^2$, it becomes a bijection.

Example 2.4

$f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = 2n$. One-one? onto?

Solution. $f(n_1) = f(n_2) \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$, so f is one-one. But odd integers such as 1 are not of the form $2n$, so f is not onto.

Example 2.5

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3$. Show f is a bijection.

Solution. $x_1^3 = x_2^3 \Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$, and the second factor is positive unless both are 0, so $x_1 = x_2$. For any $y \in \mathbb{R}$, $x = \sqrt[3]{y}$ gives $f(x) = y$. Hence f is bijective.

Counting Functions between Finite Sets

If $n(A) = m$ and $n(B) = n$	Number
Total functions $A \rightarrow B$	n^m
One-one functions $A \rightarrow B$ ($m \leq n$)	$n(n-1)(n-2)\dots(n-m+1)$
Bijections $A \rightarrow B$ ($m = n$)	$n!$

Theorem

Let A and B be finite sets with the same number of elements and $f : A \rightarrow B$. Then f is one-one $\Leftrightarrow f$ is onto $\Leftrightarrow f$ is bijective.

This result fails for infinite sets: $f(n) = 2n$ on $\mathbb{Z}$ is one-one but not onto.

2.3 Composition of Functions

Definition: Composite Function

Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The **composition** $g \circ f : A \rightarrow C$ is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Note the order: f is applied first and g second. The composition $g \circ f$ is defined only when the range of f lies in the domain of g .

Example 2.6

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x + 1$ and $g(x) = x^2$. Find $g \circ f$ and $f \circ g$.

Solution. $(g \circ f)(x) = g(x + 1) = (x + 1)^2$. $(f \circ g)(x) = f(x^2) = x^2 + 1$.

At $x = 2$: $(g \circ f)(2) = 9$ while $(f \circ g)(2) = 5$. Hence **composition is not commutative**.

Example 2.7

Let $f(x) = 2x + 1$ and $g(x) = x^2 - 3$, both $\mathbb{R} \rightarrow \mathbb{R}$. Find $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \circ f)(x)$.

Solution. $(f \circ g)(x) = 2(x^2 - 3) + 1 = 2x^2 - 5$. $(g \circ f)(x) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$. $(f \circ f)(x) = 2(2x + 1) + 1 = 4x + 3$.

Properties of Composition

Theorem: Associativity

If $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. For $x \in A$: $[h \circ (g \circ f)](x) = h((g \circ f)(x)) = h(g(f(x)))$, and $[(h \circ g) \circ f](x) = (h \circ g)(f(x)) = h(g(f(x)))$. Both sides agree for every x , so the functions are equal. ■

Theorem

Let $f : A \rightarrow B$ and $g : B \rightarrow C$.

- (i) If f and g are one-one, then $g \circ f$ is one-one.
- (ii) If f and g are onto, then $g \circ f$ is onto.
- (iii) If f and g are bijections, then $g \circ f$ is a bijection.

Proof of (i). Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$, i.e. $g(f(x_1)) = g(f(x_2))$. Since g is one-one, $f(x_1) = f(x_2)$; since f is one-one, $x_1 = x_2$. ■

Proof of (ii). Let $z \in C$. As g is onto, there is $y \in B$ with $g(y) = z$. As f is onto, there is $x \in A$ with $f(x) = y$. Then $(g \circ f)(x) = g(y) = z$. ■

Remember

If $g \circ f$ is one-one then f is one-one. If $g \circ f$ is onto then g is onto. (The converses of (i) and (ii) are not true in general.)

For any $f : A \rightarrow B$ we have $f \circ I_A = f$ and $I_B \circ f = f$.

2.4 Inverse of a Function

Definition: Invertible Function

$f : A \rightarrow B$ is **invertible** if there exists $g : B \rightarrow A$ such that $g \circ f = I_A$ and $f \circ g = I_B$. Such g is unique and is denoted f^{-1} .

Theorem

A function $f : A \rightarrow B$ is invertible if and only if it is a bijection.

Proof. ($\Rightarrow$) Let $g = f^{-1}$. If $f(x_1) = f(x_2)$, applying g gives $x_1 = x_2$, so f is one-one. For $y \in B$, $x = g(y)$ satisfies $f(x) = f(g(y)) = y$, so f is onto.

($\Leftarrow$) Let f be a bijection. For $y \in B$, onto gives some x with $f(x) = y$, and one-one makes this x unique. Define $g(y) = x$. Then $f(g(y)) = y$ and $g(f(x)) = x$, so $g = f^{-1}$. ■

Theorem: Inverse of a Composite

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Also $(f^{-1})^{-1} = f$.

Proof. $(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A$, and similarly $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$. ■

Example 2.8

Find the inverse of $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x + 3$.

Solution. f is a bijection (Example 2.2). Put $y = 2x + 3$, so $x = (y - 3)/2$. Hence $f^{-1}(y) = (y - 3)/2$, i.e. $f^{-1}(x) = (x - 3)/2$.

Check: $f(f^{-1}(x)) = 2 \cdot (x - 3)/2 + 3 = x$. ✓

Example 2.9

Let $f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{1\}$, $f(x) = (x - 1)/(x - 2)$. Show f is a bijection and find f^{-1} .

Solution. Let $y = (x - 1)/(x - 2)$. Then $y(x - 2) = x - 1$, so $x(y - 1) = 2y - 1$, giving $x = (2y - 1)/(y - 1)$. For $y \neq 1$ this is defined and $x \neq 2$ (since $x = 2$ would give $2y - 1 = 2y - 2$, impossible). So each y has exactly one pre-image, hence f is a bijection and $f^{-1}(y) = (2y - 1)/(y - 1)$.

2.5 Equivalence Classes and Partitions

Definition: Equivalence Class

Let $\sim$ be an equivalence relation on A . For $a \in A$, the **equivalence class** of a is $[a] = \{x \in A : x \sim a\}$, the set of all elements related to a .

Definition: Partition

A **partition** of a non-empty set A is a collection of non-empty subsets $\{A_i\}$ of A such that (i) the union of all A_i equals A , and (ii) $A_i \cap A_j = \emptyset$ for $i \neq j$ (the subsets are pairwise disjoint).

Theorem: Properties of Equivalence Classes

Let $\sim$ be an equivalence relation on A and $a, b \in A$. Then:

- (i) $a \in [a]$, so $[a] \neq \emptyset$.
- (ii) $a \sim b \Leftrightarrow [a] = [b]$.
- (iii) Either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Proof. (i) By reflexivity $a \sim a$. (ii) Let $a \sim b$ and $x \in [a]$. Then $x \sim a$ and $a \sim b$, so $x \sim b$ by transitivity, i.e. $x \in [b]$. Hence $[a] \subseteq [b]$. By symmetry $b \sim a$, so similarly $[b] \subseteq [a]$. Conversely, $[a] = [b]$ gives $a \in [b]$, i.e. $a \sim b$. (iii) If $x \in [a] \cap [b]$, then $x \sim a$ and $x \sim b$, so $a \sim x \sim b$ gives $a \sim b$ and by (ii) $[a] = [b]$. ■

Theorem: Fundamental Theorem on Equivalence Relations

- (a) An equivalence relation on A produces a partition of A into its distinct equivalence classes.
- (b) Conversely, every partition of A defines an equivalence relation on A ($x \sim y \Leftrightarrow x$ and y lie in the same block).

Thus equivalence relations on A and partitions of A are two descriptions of the same thing. The set of all equivalence classes is called the **quotient set** $A/\sim$.

Example 2.10

Find the equivalence classes of congruence modulo 3 on $\mathbb{Z}$.

Solution. $[0] = \{\dots, -6, -3, 0, 3, 6, \dots\}$, $[1] = \{\dots, -5, -2, 1, 4, 7, \dots\}$, $[2] = \{\dots, -4, -1, 2, 5, 8, \dots\}$. These three classes are disjoint and their union is $\mathbb{Z}$, so they form a partition. The quotient set is $\mathbb{Z}_3 = \{[0], [1], [2]\}$. In general $\mathbb{Z}_n$ has n classes.

Example 2.11

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(a, b) : a \equiv b \pmod{3}\}$. Find the classes.

Solution. $[1] = [4] = \{1, 4\}$, $[2] = [5] = \{2, 5\}$, $[3] = [6] = \{3, 6\}$. Partition: $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$.

2.6 Summary of Module II

- Function = every input has exactly one output.
- One-one: distinct inputs, distinct outputs. Onto: every codomain element is hit. Bijection = both.
- $g \circ f$ means “first f , then g ”; composition is associative but not commutative.
- f is invertible $\Leftrightarrow f$ is a bijection; $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- Equivalence relations and partitions correspond to each other.

2.7 Practice Questions: Module II

1. Define one-one, onto and bijective functions with examples.
2. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 5x - 7$ is a bijection and find f^{-1} . (Ans: $f^{-1}(x) = (x + 7)/5$)
3. Is $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = n + 1$ one-one? onto? (Ans: one-one, not onto)
4. If $f(x) = x + 2$ and $g(x) = 3x$, find $g \circ f$ and $f \circ g$. (Ans: $3x + 6$, $3x + 2$)
5. Prove that composition of functions is associative.
6. Prove that a function is invertible if and only if it is a bijection.
7. Prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
8. Define equivalence class and prove that two equivalence classes are either equal or disjoint.
9. Find the equivalence classes of congruence modulo 4 on $\mathbb{Z}$.
10. If $n(A) = 4$ and $n(B) = 3$, find the number of functions from A to B and from B to A. (Ans: 81 and 64)

MODULE III: Elementary Operations and Rank of a Matrix

Lectures: 07 | Text Book 2: Chapter 11 (Articles 11.5 to 11.16)

3.1 Matrices: Quick Recap

A **matrix** of order $m \times n$ is a rectangular array of mn numbers arranged in m rows and n columns. We write $A = [a_{ij}]_{m \times n}$, where a_{ij} is the entry in the i -th row and j -th column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

A matrix of order 2×3

Type	Description
Row / Column matrix	Only one row ($1 \times n$) / only one column ($m \times 1$)
Square matrix	Number of rows = number of columns ($n \times n$)
Zero matrix	All entries are 0
Diagonal matrix	Square; all non-diagonal entries are 0
Identity matrix I_n	Diagonal matrix with every diagonal entry 1
Upper / Lower triangular	All entries below / above the main diagonal are 0
Symmetric / Skew-symmetric	$A^T = A$ / $A^T = -A$
Singular / Non-singular	$\det A = 0$ / $\det A \neq 0$

3.2 Elementary Operations

Definition: Elementary Operations

An **elementary operation** is any one of the following three operations on the rows (or columns) of a matrix:

Type 1 (Interchange): interchange of the i -th and j -th rows, written $R_i \leftrightarrow R_j$ (columns: $C_i \leftrightarrow C_j$).

Type 2 (Scaling): multiplication of the i -th row by a non-zero scalar k , written $R_i \rightarrow kR_i$ (columns: $C_i \rightarrow kC_i$).

Type 3 (Addition): adding k times the j -th row to the i -th row, written $R_i \rightarrow R_i + kR_j$ (columns: $C_i \rightarrow C_i + kC_j$).

Operations on rows are called **elementary row operations**; operations on columns are **elementary column operations**. Each elementary operation is reversible: $R_i \leftrightarrow R_j$ is its own inverse, $R_i \rightarrow kR_i$ is undone by $R_i \rightarrow (1/k)R_i$, and $R_i \rightarrow R_i + kR_j$ is undone by $R_i \rightarrow R_i - kR_j$.

Elementary Matrices

Definition: Elementary Matrix

A matrix obtained from the identity matrix I_n by a **single** elementary operation is called an **elementary matrix**.

Example: for 3×3 matrices, the elementary matrices corresponding to $R_1 \leftrightarrow R_2$, $R_2 \rightarrow 5R_2$ and $R_3 \rightarrow R_3 + 2R_1$ are:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

Theorem

Performing an elementary row operation on A gives the same result as pre-multiplying A by the corresponding elementary matrix. Performing an elementary column operation on A is the same as post-multiplying A by the corresponding elementary matrix.

Every elementary matrix is non-singular, and its inverse is again an elementary matrix of the same type.

3.3 Equivalent Matrices

Definition: Equivalent Matrices

A matrix B is said to be **equivalent** to A , written $A \sim B$, if B can be obtained from A by a finite sequence of elementary row and column operations.

Equivalent matrices have the same order and the same rank. Equivalence of matrices is an equivalence relation (reflexive, symmetric and transitive), because the inverse and the composition of elementary operations are again elementary operations. Two matrices A and B of the same order are equivalent if and only if $B = PAQ$ for some non-singular matrices P and Q .

Normal (Canonical) Form

Theorem: Normal Form

Every $m \times n$ matrix of rank r is equivalent to a matrix of the form

$$[I_r \ 0; 0 \ 0]$$

where I_r is the $r \times r$ identity matrix and 0 denotes zero blocks of suitable size. This is called the **normal form** of A .

Example 3.1

Reduce A to normal form and find its rank, where A is shown below.

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{array} \right|$$

Solution. Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$:

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{array} \right|$$

Apply $R_3 \rightarrow R_3 - 2R_2$:

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{array} \right|$$

Apply $C_2 \rightarrow C_2 - 2C_1$, $C_3 \rightarrow C_3 - 3C_1$:

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{array} \right|$$

Apply $C_3 \rightarrow C_3 - 2C_2$ and then $R_2 \rightarrow (-1)R_2$:

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right|$$

This is the normal form $[I_2 \ 0; \ 0 \ 0]$, so **rank A = 2**.

3.4 Echelon Form

Definition: Row Echelon Form (REF)

A matrix is in **row echelon form** if:

- (i) all zero rows (if any) are at the bottom;
- (ii) the first non-zero entry (the **leading entry** or **pivot**) of each non-zero row lies strictly to the right of the leading entry of the row above it;
- (iii) (commonly) all entries below each leading entry are 0.

Example 3.2

Which of these is in echelon form?

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{array} \right|$$

(a) Yes: staircase pattern, entries below pivots are 0.

$$\left| \begin{array}{ccc} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{array} \right|$$

(b) No: the pivot of row 2 lies to the left of the pivot of row 1.

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right|$$

(c) No: the zero row is not at the bottom.

Procedure to Reduce a Matrix to Echelon Form

1. Locate the leftmost non-zero column; bring a non-zero entry to the top of it by interchange if required (this is the first pivot).
2. Use Type 3 operations to make all entries below the pivot zero.
3. Ignore the first row and the pivot column; repeat the process on the remaining sub-matrix.
4. Stop when no non-zero rows remain below the current row.

Example 3.3

Reduce A to echelon form, where

$$\left| \begin{array}{cccc} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{array} \right|$$

Solution. $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$ give:

$$\left| \begin{array}{cccc} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{array} \right|$$

$R_3 \rightarrow R_3 - 2R_2$ gives the echelon form:

$$\left| \begin{array}{cccc} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{array} \right|$$

The number of non-zero rows is 2, hence rank A = 2.

3.5 Row Reduced Echelon Form (RREF)

Definition: Row Reduced Echelon Form

A matrix is in **row reduced echelon form** (RREF) if:

- (i) it is in row echelon form;
- (ii) the leading entry of each non-zero row is **1**;
- (iii) each leading 1 is the **only non-zero entry** in its column.

Theorem: Uniqueness of RREF

Every matrix can be reduced by elementary row operations to one and only one RREF. (A matrix may have many echelon forms, but only one reduced echelon form.)

Procedure: Gauss–Jordan Elimination

1. Reduce the matrix to echelon form using the procedure above.
2. Divide each non-zero row by its leading entry so that every pivot becomes 1.
3. Starting from the last pivot and moving upward, use Type 3 operations to make all entries above each pivot zero.

Example 3.4

Find the RREF of A, where

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{array} \right|$$

Solution. $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$:

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{array} \right|$$

$R_3 \rightarrow R_3 - 3R_2$ (this is an echelon form):

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right|$$

$R_3 \rightarrow \frac{1}{2}R_3$:

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right|$$

$R_2 \rightarrow R_2 - 2R_3$ and $R_1 \rightarrow R_1 - R_3$:

$$\left| \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right|$$

$R_1 \rightarrow R_1 - R_2$:

$$\left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right|$$

The RREF is the identity matrix I_3 , so rank A = 3.

Example 3.5

Find the RREF of A, where

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 9 \\ 3 & 6 & 10 & 13 \end{vmatrix}$$

Solution. $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$R_3 \rightarrow R_3 - R_2$ (echelon form reached):

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$R_1 \rightarrow R_1 - 3R_2$ gives the RREF:

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

The pivots are in columns 1 and 3; the number of pivots (non-zero rows) is 2, so $\text{rank } A = 2$.

3.6 Row Rank, Column Rank and Rank of a Matrix

Definition: Row Rank and Column Rank

The **row rank** of A is the maximum number of linearly independent rows of A. The **column rank** of A is the maximum number of linearly independent columns of A.

Theorem

For every matrix, row rank = column rank. This common value is called the **rank** of the matrix, denoted $p(A)$ or $\text{rank}(A)$.

Definition: Rank (minor definition)

The **rank** of an $m \times n$ matrix A is the largest integer r such that A has at least one non-zero minor (determinant of a square sub-matrix) of order r , while every minor of order greater than r is zero.

Important Properties of Rank

- Rank of a zero matrix is 0; the rank of any non-zero matrix is at least 1.
- For an $m \times n$ matrix, $0 \leq \text{rank}(A) \leq \min(m, n)$.
- Rank of I_n is n . An $n \times n$ matrix is non-singular $\Leftrightarrow \text{rank} = n$.

- Rank is unchanged by elementary row and column operations (equivalent matrices have equal rank).
- $\text{rank}(A) = \text{rank}(A^T)$.
- $\text{rank}(AB) \leq \min\{\text{rank } A, \text{rank } B\}$.
- Rank of a matrix in echelon form = number of non-zero rows.

Methods of Finding Rank

- Minor method:** find the highest-order non-vanishing minor.
- Echelon method (recommended):** reduce to row echelon form using row operations; rank = number of non-zero rows.
- Normal form method:** reduce to $[I_r \ 0 \ ; \ 0 \ 0]$; rank = r .

Example 3.6

Find the rank of A by the minor method, where

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix}$$

Solution. $\det A = 0$, because $R_2 = 2R_1$. So rank < 3 . All 2×2 minors vanish as well (for example $1 \cdot 4 - 2 \cdot 2 = 0$, $1 \cdot 6 - 3 \cdot 2 = 0$). But A has non-zero entries, so there is a non-zero minor of order 1. Hence **rank A = 1**.

Example 3.7

Find the rank of A, where

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix}$$

Solution. $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$, $R_4 \rightarrow R_4 - 4R_1$:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{vmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$ and $R_4 \rightarrow R_4 - 3R_2$:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Two non-zero rows remain, so **rank A = 2**.

Example 3.8

Find the value of k for which the rank of the matrix below is 2.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & k \end{vmatrix}$$

Solution. $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ give $R_2 = (0, 0, 0)$ and $R_3 = (0, 0, k - 9)$. If $k = 9$ the rank is 1; if $k \neq 9$ there are two non-zero rows (the first and the third), so the rank is 2. Hence rank = 2 for **every** $k \neq 9$.

Example 3.9

Find the rank of A , where

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$

Solution. $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$:

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$R_3 \rightarrow R_3 - R_2$:

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

Three non-zero rows, so **rank $A = 3$** (A is non-singular, $\det A = 1$).

Remember

Always put a row operation next to each step. Never apply a row operation and a column operation together in the same step when finding the rank by the echelon method; use only row operations (or only column operations) to be safe.

3.7 Summary of Module III

- Three elementary operations: interchange, scaling by a non-zero constant, adding a multiple of one row to another.
- Equivalent matrices are connected by elementary operations and have equal rank.
- REF: staircase of pivots with zero rows at the bottom. RREF: pivots equal to 1 and the only non-zero entry of their column (unique).
- Rank = number of non-zero rows of the echelon form = largest order of a non-zero minor.

3.8 Practice Questions: Module III

1. Define the three types of elementary operations with notation.
2. Define the rank of a matrix. State any four properties.
3. Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 4 \end{bmatrix}$. (Ans: 2)
4. Reduce $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}$ to echelon form and find its rank. (Ans: 2)
5. Find the RREF of $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$. (Ans: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, rank 2)
6. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$. (Ans: 1)
7. Reduce $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ to normal form.
8. Distinguish between echelon form and row reduced echelon form with examples.
9. Find the values of k for which the rank of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$ is less than 3. (Ans: $k = 1$ or $k = 2$; the determinant equals $(k - 1)(k - 2)$)

MODULE IV: Systems of Linear Equations and the Cayley–Hamilton Theorem

Lectures: 08 | Text Book 2: Chapter 11 (Articles 11.32 to 11.39) and Chapter 12 (Articles 12.1 to 12.3, 12.18)

4.1 System of Linear Equations

A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ has the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

In matrix form this is $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = [a_{ij}]$ is the $m \times n$ **coefficient matrix**, $\mathbf{X} = (x_1, \dots, x_n)^T$ is the column of unknowns and $\mathbf{B} = (b_1, \dots, b_m)^T$ is the column of constants. The matrix $[\mathbf{A} \mid \mathbf{B}]$ of order $m \times (n + 1)$ is the **augmented matrix**.

Definition: Homogeneous and Non-Homogeneous Systems

The system $\mathbf{AX} = \mathbf{B}$ is **non-homogeneous** if $\mathbf{B} \neq \mathbf{0}$ and **homogeneous** if $\mathbf{B} = \mathbf{0}$, i.e. $\mathbf{AX} = \mathbf{0}$.

Definition: Solution and Consistency

A **solution** is a set of values of $x_1, \dots, x_n$ satisfying all the equations. A system is **consistent** if it has at least one solution and **inconsistent** if it has none.

4.2 Non-Homogeneous Systems: Consistency Theorem

Theorem: Rouché–Capelli (Consistency) Theorem

Let $\mathbf{AX} = \mathbf{B}$ be a system of m equations in n unknowns. Then:

- (i) The system is **consistent** if and only if $\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \mid \mathbf{B}]$.
- (ii) If $\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \mid \mathbf{B}] = n$ (number of unknowns), the solution is **unique**.
- (iii) If $\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \mid \mathbf{B}] = r < n$, there are **infinitely many** solutions, involving $n - r$ arbitrary parameters.
- (iv) If $\text{rank } \mathbf{A} \neq \text{rank } [\mathbf{A} \mid \mathbf{B}]$, the system is **inconsistent** (no solution).

Condition	Conclusion
$\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \mid \mathbf{B}] = n$	Unique solution
$\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \mid \mathbf{B}] < n$	Infinitely many solutions ($n - r$ free variables)
$\text{rank } \mathbf{A} < \text{rank } [\mathbf{A} \mid \mathbf{B}]$	No solution (inconsistent)

Method of Solving (Gauss Elimination)

1. Write the augmented matrix $[\mathbf{A} \mid \mathbf{B}]$.

2. Apply row operations only to reduce it to echelon form.
3. Compare rank A and rank $[A \mid B]$ and apply the consistency theorem.
4. If consistent, write the equations from the echelon form and solve by back-substitution.

Example 4.1

Solve: $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$.

Solution. The augmented matrix is:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{array} \right|$$

$R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -1 & -3 & -11 \end{array} \right|$$

$R_2 \rightarrow (-\frac{1}{2})R_2$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -3 & -11 \end{array} \right|$$

$R_3 \rightarrow R_3 + R_2$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{array} \right|$$

Here rank A = rank $[A \mid B] = 3$ = number of unknowns, so the solution is unique. Back-substitution: $-3z = -9 \Rightarrow z = 3$; $y = 2$; $x + 2 + 3 = 6 \Rightarrow x = 1$.

Answer: $x = 1$, $y = 2$, $z = 3$. Check in the third equation: $2(1) + 2 - 3 = 1 \checkmark$.

Example 4.2

Test the consistency of: $x + y + z = 3$, $x + 2y + 3z = 4$, $2x + 3y + 4z = 9$.

Solution. Augmented matrix, then $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array} \right|$$

$R_3 \rightarrow R_3 - R_2$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right|$$

The last row reads $0 = 2$, which is impossible. Here rank A = 2 but rank $[A \mid B] = 3$. Since the ranks differ, the system is **inconsistent** (no solution).

Example 4.3

Solve: $x + y + z = 3$, $x + 2y + 3z = 6$, $2x + 3y + 4z = 9$.

Solution. Apply $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array} \right|$$

$R_3 \rightarrow R_3 - R_2$:

$$\left| \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right|$$

$\text{rank } A = \text{rank } [A \mid B] = 2 < 3 = n$, so there are infinitely many solutions with $n - r = 1$ free parameter. Put $z = k$. Then $y + 2k = 3 \Rightarrow y = 3 - 2k$, and $x = 3 - y - z = 3 - (3 - 2k) - k = k$.

Solution set: $(x, y, z) = (k, 3 - 2k, k)$, $k \in \mathbb{R}$. For $k = 1$ we get $(1, 1, 1)$.

4.3 Homogeneous Systems

A homogeneous system $AX = 0$ is always consistent, because $X = 0$ ($x_1 = x_2 = \dots = x_n = 0$), called the **trivial solution**, always satisfies it. The question is whether **non-trivial** solutions exist.

Theorem

Let $AX = 0$ be a homogeneous system of m equations in n unknowns.

- (i) If $\text{rank } A = n$, the system has only the trivial solution.
- (ii) If $\text{rank } A = r < n$, the system has infinitely many non-trivial solutions, with $n - r$ free parameters.
- (iii) For a square system ($m = n$): non-trivial solutions exist $\Leftrightarrow \det A = 0$.
- (iv) If $m < n$ (fewer equations than unknowns), non-trivial solutions always exist.

Example 4.4

Solve: $x + 2y + z = 0$, $2x + y + 3z = 0$, $3x + 3y + 4z = 0$.

Solution. Coefficient matrix and $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$:

$$\left| \begin{array}{ccc} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \end{array} \right|$$

$R_3 \rightarrow R_3 - R_2$:

$$\left| \begin{array}{ccc} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{array} \right|$$

$\text{rank } A = 2 < 3$, so non-trivial solutions exist with $3 - 2 = 1$ parameter. From $-3y + z = 0$ we get $z = 3y$. Put $y = k$: $z = 3k$, and $x = -2y - z = -2k - 3k = -5k$.

Solution: $(x, y, z) = (-5k, k, 3k), k \in \mathbb{R}$.

Example 4.5

Show that $x + y + z = 0, x - y + z = 0, 2x + y - z = 0$ has only the trivial solution.

Solution. The determinant of the coefficient matrix is $1(1 - 1) - 1(-1 - 2) + 1(1 + 2) = 0 + 3 + 3 = 6 \neq 0$. So rank $A = 3 = n$ and the only solution is $x = y = z = 0$.

Example 4.6

For what value of λ do the equations $x + y + z = 0, x + 2y + 3z = 0, x + 3y + \lambda z = 0$ have non-trivial solutions?

Solution. Need $\det = 0$: $1(2\lambda - 9) - 1(\lambda - 3) + 1(3 - 2) = 2\lambda - 9 - \lambda + 3 + 1 = \lambda - 5$. So $\lambda = 5$.

4.4 Characteristic Values and Characteristic Vectors

Definition: Characteristic Matrix, Polynomial and Equation

Let A be an $n \times n$ matrix. The matrix $A - \lambda I$ is the **characteristic matrix** of A . The determinant $|A - \lambda I|$ is a polynomial of degree n in λ , called the **characteristic polynomial**. The equation $|A - \lambda I| = 0$ is the **characteristic equation**.

Definition: Characteristic Values and Vectors

The roots λ of the characteristic equation are the **characteristic values** (eigenvalues, latent roots) of A . For a characteristic value λ , a non-zero vector X satisfying

$$AX = \lambda X \text{ (equivalently } (A - \lambda I)X = 0)$$

is called a **characteristic vector** (eigenvector) of A corresponding to λ .

For a 3×3 matrix the characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = \text{trace of } A$ (sum of diagonal elements) and $S_2 = \text{sum of the principal minors of order 2}$ (the 2×2 minors obtained by deleting row i and column i). For a 2×2 matrix: $\lambda^2 - (\text{trace } A)\lambda + |A| = 0$.

Properties

- The sum of the characteristic values = trace of A ; the product = $|A|$.
- A is singular $\Leftrightarrow 0$ is a characteristic value of A .
- A and A^T have the same characteristic values.
- The characteristic values of a triangular (or diagonal) matrix are its diagonal entries.
- If λ is a characteristic value of A then λ^k is one of A^k , and $1/\lambda$ is one of A^{-1} ($\lambda \neq 0$).
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Procedure

1. Form $|A - \lambda I| = 0$ and solve for λ .

2. For each λ , solve $(A - \lambda I)X = 0$ (a homogeneous system); non-zero solutions are the eigenvectors.

Example 4.7

Find the characteristic values and vectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution. $|A - \lambda I| = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$, so $\lambda = 1, 3$. (Check: sum = 4 = trace, product = 3 = $|A|$.)

For $\lambda = 1$: $(A - I)X = 0$ gives $x + y = 0$, so $X = k(1, -1)^T$.

For $\lambda = 3$: $(A - 3I)X = 0$ gives $-x + y = 0$, so $X = k(1, 1)^T$.

Example 4.8

Find the characteristic values and vectors of A below.

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{vmatrix}$$

Solution. $|A - \lambda I| = (2 - \lambda)[(3 - \lambda)(9 - \lambda) - 16] = (2 - \lambda)(\lambda^2 - 12\lambda + 11) = (2 - \lambda)(\lambda - 1)(\lambda - 11)$. So $\lambda = 2, 1, 11$.

$\lambda = 2$: $(A - 2I)X = 0$ gives $y + 4z = 0$ and $4y + 7z = 0$, so $y = z = 0$ and $X = k(1, 0, 0)^T$.

$\lambda = 1$: $x = 0$ and $2y + 4z = 0$, so $y = -2z$ and $X = k(0, -2, 1)^T$.

$\lambda = 11$: $x = 0$ and $-8y + 4z = 0$, so $z = 2y$ and $X = k(0, 1, 2)^T$.

4.5 Cayley–Hamilton Theorem

Theorem: Cayley–Hamilton Theorem

Every square matrix satisfies its own characteristic equation. That is, if the characteristic polynomial of A is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n,$$

then $A^n + a_1A^{n-1} + \dots + a_{n-1}A + a_nI = O$ (the zero matrix).

Idea of proof (2×2 case). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc)$. Direct computation gives $A^2 = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix}$. Then $A^2 - (a + d)A + (ad - bc)I$ has entries $a^2 + bc - a^2 - ad + ad - bc = 0$ in the (1,1) place, $b(a + d) - b(a + d) = 0$ in the (1,2) place, and similarly 0 in the other two. Thus the result is the zero matrix. The general proof uses the adjoint matrix: $(A - \lambda I) \text{adj}(A - \lambda I) = p(\lambda)I$, and comparing coefficients of powers of λ .

Applications

- **Finding the inverse** of a non-singular matrix ($|A| \neq 0$): multiply the relation by A^{-1} .
- **Finding higher powers** of A (such as A^4, A^5) by reducing them using the relation.
- **Evaluating matrix polynomials** such as $A^5 - 3A^4 + A^2 - 5I$.

Example 4.9

Verify the Cayley–Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and find A^{-1} .

Solution. Characteristic equation: $\lambda^2 - 5\lambda - 2 = 0$ (trace 5, determinant $4 - 6 = -2$).

$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Then $A^2 - 5A - 2I = \begin{bmatrix} 7 - 5 - 2 & 10 - 10 \\ 15 - 15 & 22 - 20 - 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$. ✓
Verified.

Inverse: $A^2 - 5A = 2I \Rightarrow A(A - 5I) = 2I \Rightarrow A^{-1} = \frac{1}{2}(A - 5I) = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

Check: first row of A times first column of $A^{-1} = 1(-2) + 2(3/2) = 1$; first row times second column $= 1(1) + 2(-1/2) = 0$ ✓.

Example 4.10

Verify the Cayley–Hamilton theorem for A below, and hence find A^{-1} .

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix}$$

Solution. Trace $= 1 + 2 + 3 = 6$. The principal 2×2 minors: deleting row/column 1: $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$; deleting 2: $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$; deleting 3: $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$. Sum $S_2 = 7$. Determinant $= 1(6 - 0) - 0 + 2(0 - 4) = -2$. Hence

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$$

We compute the powers of A :

$$\begin{vmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{vmatrix}$$

A^2

$$\begin{vmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{vmatrix}$$

A^3

Now $A^3 - 6A^2 + 7A + 2I$: the (1,1) entry is $21 - 30 + 7 + 2 = 0$; the (1,3) entry is $34 - 48 + 14 = 0$; the (2,2) entry is $8 - 24 + 14 + 2 = 0$; the (2,3) entry is $23 - 30 + 7 = 0$; the (3,3) entry is $55 - 78 + 21 + 2 = 0$; all remaining entries are likewise 0. So $A^3 - 6A^2 + 7A + 2I = O$. ✓

Inverse. From $A^3 - 6A^2 + 7A = -2I$ we get $A(A^2 - 6A + 7I) = -2I$, so $A^{-1} = -\frac{1}{2}(A^2 - 6A + 7I)$. Now $A^2 - 6A + 7I =$

$$\begin{vmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{vmatrix}$$

Therefore

$$\begin{vmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{vmatrix}$$

$$A^{-1}$$

Example 4.11

Using the Cayley–Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ express A^3 and A^4 in the form $pA + qI$.

Solution. Characteristic equation: $\lambda^2 - 2\lambda - 3 = 0$. So $A^2 = 2A + 3I$. Then $A^3 = 2A^2 + 3A = 2(2A + 3I) + 3A = 7A + 6I$. And $A^4 = 7A^2 + 6A = 7(2A + 3I) + 6A = 20A + 21I$.

Remember

Cayley–Hamilton lets you replace any high power of A by a lower-degree combination of A , A^2 , ..., A^{n-1} and I . Always write the characteristic equation carefully first; every later step depends on it.

4.6 Summary of Module IV

- $AX = B$ is consistent iff $\text{rank } A = \text{rank } [A \mid B]$; unique solution if this rank equals n , infinite solutions otherwise (if less than n).
- $AX = 0$ always has the trivial solution; a non-trivial solution exists iff $\text{rank } A < n$.
- Eigenvalues are the roots of $|A - \lambda I| = 0$; eigenvectors solve $(A - \lambda I)X = 0$.
- Cayley–Hamilton: every matrix satisfies its own characteristic equation; used for inverse and powers.

4.7 Practice Questions: Module IV

1. State the condition for consistency of a non-homogeneous system $AX = B$.
2. Solve $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x + y - z = 0$. (Ans: $x = 1$, $y = 3$, $z = 5$)
3. Test for consistency: $x + 2y = 3$, $2x + 4y = 7$. (Ans: inconsistent)
4. Solve $x + y + z = 0$, $x - y + z = 0$, $x + 2y - z = 0$. (Ans: trivial solution only)
5. Find the value of k for which $x + y + z = 0$, $x + 2y + 3z = 0$, $x + 4y + kz = 0$ has non-trivial solutions. (Ans: $k = 7$)
6. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. (Ans: $\lambda = 3, 2$)
7. State and verify the Cayley–Hamilton theorem for $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. (Characteristic equation: $\lambda^2 - 4\lambda + 1 = 0$)
8. Use the Cayley–Hamilton theorem to find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. (Ans: $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$)
9. Find A^{-1} using Cayley–Hamilton for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and compare with Example 4.10.
10. Distinguish between a homogeneous and a non-homogeneous system of linear equations.

Quick Revision Sheet

Topic	Key Points
Power set	$n(P(A)) = 2^n$

Topic	Key Points
Cartesian product	$n(A \times B) = n(A) \cdot n(B)$; $A \times B \neq B \times A$
Union formula	$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
Number of relations $A \rightarrow B$	$2^{(mn)}$
Equivalence relation	Reflexive + Symmetric + Transitive
Bijection	One-one + Onto; inverse exists iff bijection
Composite	$(g \circ f)(x) = g(f(x))$; $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
Number of functions	n^m total; $n!$ bijections ($m = n$)
Elementary operations	$R_i \leftrightarrow R_j$; $R_i \rightarrow kR_i$ ($k \neq 0$); $R_i \rightarrow R_i + kR_j$
Rank	Number of non-zero rows in echelon form
RREF	Pivots = 1, only non-zero entry in their column; unique
Consistency	$\text{rank } A = \text{rank } [A \mid B]$
Homogeneous system	Non-trivial solution $\Leftrightarrow \text{rank } A < n$ ($\det A = 0$ if square)
Characteristic equation	$ A - \lambda I = 0$; sum of roots = trace, product = $ A $
Cayley–Hamilton	$p(A) = O$; use to find A^{-1} and A^n

Best wishes for your examination!